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# Vector coherent state representations, induced representations and geometric quantization: II. Vector coherent state representations

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## Abstract

It is shown here and in the preceding paper (Bartlett S D, Rowe D J and Repka J 2002 *J. Phys. A: Math. Gen.* **35**) that vector coherent state theory, the theory of induced representations and geometric quantization provide alternative but equivalent quantizations of an algebraic model. The relationships are useful because some constructions are simpler and more natural from one perspective than another. More importantly, each approach suggests ways of generalizing its counterparts. In this paper, we focus on the construction of quantum models for algebraic systems with intrinsic degrees of freedom. Semi-classical partial quantizations, for which only the intrinsic degrees of freedom are quantized, arise naturally out of this construction. The quantization of the  $SU(3)$  and rigid rotor models are considered as examples.

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## 1. Introduction

Quantizing a classical model is a difficult problem in general. The theory of geometric quantization (GQ) [2] provides a general and powerful framework for the quantization of a wide variety of classical systems, but due to its formidable mathematical language it is inaccessible to most physicists. We show here and in the preceding paper [1] that the useful and physically-motivated theory of coherent state representations [3, 4] provides a natural language for describing the techniques of GQ. In [1], it was shown that scalar coherent state theory yields three categories of representations for the spectrum generating algebra (SGA) of an algebraic model: classical realizations, prequantization and the irreducible representations

of quantization. This paper generalizes the results of [1] to vector-valued coherent state representations.

While it is often possible to induce representations of a Lie algebra from a one-dimensional irrep of some subalgebra (as in the standard coherent state construction), it is generally more economical and effective to induce from a known multi-dimensional irrep of a larger subalgebra. The amount of work is then minimized by capitalizing on information that is already available, and leads to a useful physical interpretation of some degrees of freedom of a model system as intrinsic. For example, using the method of induced representations, Wigner [5] found irreps of the Poincaré group corresponding to quantizations of particles with intrinsic spin. Such intrinsic degrees of freedom are often regarded as having quantal origins. It will be seen that they have classical counterparts and that the general theory of induced representations, when developed within the framework of vector coherent state (VCS) theory [6, 7], has a natural expression in the language of geometric quantization.

## 2. Classical representations with intrinsic degrees of freedom

Let  $T$  be an abstract (possibly projective) unitary representation of a dynamical group  $G$  on a Hilbert space  $\mathbb{H}$ . As in the scalar theory,  $T$  need not be specified precisely; it could be, for example, a regular representation, or a Weil representation on a many-particle Hilbert space. Corresponding to any normalized state  $|0\rangle \in \mathbb{H}$  there is a coadjoint orbit

$$\mathcal{O}_\rho = \{\rho_g; g \in G\} \quad (1)$$

of densities defined by

$$\rho_g(A) = \langle 0 | \hat{A}(g) | 0 \rangle \quad (2)$$

where  $\hat{A} = T(A)$  and  $\hat{A}(g) = T(g)\hat{A}T(g^{-1})$ . Let  $H_\rho \subset G$  be the isotropy subgroup of  $\mathcal{O}_\rho$  at  $\rho$ ; the orbit  $\mathcal{O}_\rho \simeq H_\rho \backslash G$  is known to be symplectic and can be regarded as a classical phase space. Moreover, a classical representation  $\mathcal{A}$  of an element  $A \in \mathfrak{g}$ , the Lie algebra of  $G$ , is given as a function on  $\mathcal{O}_\rho$  by  $\mathcal{A}(g) = \rho_g(A)$  (for details, see [1]).

Let  $H \supseteq H_\rho$  be some other subgroup. It may be convenient to choose  $H$  such that  $H \backslash G$  is also symplectic, but this condition is not necessary. The phase space  $\mathcal{O}_\rho \simeq H_\rho \backslash G$  may then be viewed as a  $H_\rho \backslash G \rightarrow H \backslash G$  fibre bundle with typical fibre  $H_\rho \backslash H$ . When  $H$  is set equal to  $H_\rho$ , as in scalar coherent state theory, the fibres become trivial. A specification of  $H$  that contains  $H_\rho$  as a proper subgroup, in vector coherent state theory, corresponds to regarding some degrees of freedom of  $G$  as intrinsic, i.e. as gauge degrees of freedom. We refer to  $H$  as the *intrinsic symmetry group*.

Viewing the classical phase space as a smaller space with intrinsic degrees of freedom in this way does not change a classical representation in principle. However, it gives a new perspective and leads to new quantization procedures. Starting with a density  $\rho \in \mathfrak{g}^*$ , the classical phase space  $\mathcal{O}_\rho$  is generated in two steps. The first step generates the coadjoint orbit  $H_\rho \backslash H$  of the subgroup  $H \subset G$  as the set of densities  $\{\rho_\alpha; \alpha \in H\}$ . This set is then regarded as the fibre of a bundle over the point  $H$  of the space  $H \backslash G$ . The second step defines the fibre over an arbitrary point  $Hg$  of  $H \backslash G$  as the set  $\{\rho_{\alpha g}; \alpha \in H\}$ . The classical function  $\mathcal{A}$  on  $G$  representing an element  $A \in \mathfrak{g}$ , defined as having values  $\mathcal{A}(g) = \rho(A(g))$ , with  $A(g) = \text{Ad}_g(A)$  ( $=gAg^{-1}$  for a matrix group) is then seen as being  $H$ -equivariant, i.e. it satisfies the equation

$$\mathcal{A}(\alpha g) = \rho_\alpha(A(g)) \quad \forall \alpha \in H. \quad (3)$$

When  $H = H_\rho$ , the fibres are trivial and this equivariance condition reduces to the invariance condition  $\mathcal{A}(\alpha g) = \mathcal{A}(g)$  for  $\alpha \in H$ .

As an example, consider a particle moving in a three-dimensional Euclidean space. If the particle has intrinsic spin, it is appropriate to take as SGA the semidirect sum of  $hw(3)$ , a Heisenberg–Weyl algebra, and  $su(2)$  with basis  $\{\hat{q}_i, \hat{p}_i, \hat{I}, \hat{J}_i; i = 1, 2, 3\}$  and commutation relations

$$\begin{aligned} [\hat{q}_i, \hat{p}_j] &= i\hbar\delta_{ij}\hat{I} & [\hat{J}_i, \hat{q}_j] &= i\hbar\sum_k \varepsilon_{ijk}\hat{q}_k \\ [\hat{J}_i, \hat{J}_j] &= i\sum_k \varepsilon_{ijk}\hat{J}_k & [\hat{J}_i, \hat{p}_j] &= i\hbar\sum_k \varepsilon_{ijk}\hat{p}_k. \end{aligned} \tag{4}$$

We suppose these Lie algebra elements act via a representation  $T$  as Hermitian operators on some Hilbert space  $\mathbb{H}$ . Let  $|0\rangle \in \mathbb{H}$  be a state with expectation values

$$\begin{aligned} \langle 0|\hat{I}|0\rangle &= 1 & \langle 0|\hat{J}_3|0\rangle &= M \\ \langle 0|\hat{q}_i|0\rangle &= \langle 0|\hat{p}_i|0\rangle = \langle 0|\hat{J}_1|0\rangle = \langle 0|\hat{J}_2|0\rangle = 0. \end{aligned} \tag{5}$$

An  $[HW(3)]SU(2)$  group element can be parameterized

$$T(g(v, q, p)) = T(v)\exp\left(-\frac{i}{\hbar}\sum_i p_i\hat{q}_i\right)\exp\left(\frac{i}{\hbar}\sum_i q_i\hat{p}_i\right) \tag{6}$$

with  $v$  a  $U(2)$  group element. For  $M \neq 0$ , the isotropy subgroup  $H_\rho$  of the density defined by  $\rho(\hat{A}) = \langle 0|\hat{A}|0\rangle$  is the group  $H_\rho \simeq U(1) \times U(1)$  with infinitesimal generators  $\{\hat{I}, \hat{J}_3\}$ . Thus, with  $\rho(\hat{A}) = \langle 0|\hat{A}|0\rangle$  and  $g = g(v, q, p)$ , the classical representation of the observables  $\{\hat{q}_i, \hat{p}_i, \hat{I}, \hat{J}_i\}$  is given by the functions  $\{\mathcal{Q}_i, \mathcal{P}_i, \mathcal{I}, \mathcal{J}_i\}$  with

$$\begin{aligned} \mathcal{Q}_i(g) &= \rho(\hat{q}_i(g)) = q_i \\ \mathcal{P}_i(g) &= \rho(\hat{p}_i(g)) = p_i \\ \mathcal{J}_i(g) &= \rho(\hat{J}_i(g)) = \mathcal{S}_i(v) + (q_j p_k - q_k p_j) \end{aligned} \tag{7}$$

where  $\mathcal{S}_i$ , a function over  $SU(2)$ , represents the intrinsic spin of the particle. The functions of this classical representation can be regarded as functions over  $H_\rho \backslash [HW(3)]SU(2)$ . However, they are more usefully represented as functions over the classical  $(p-q)$  phase space,  $U(2) \backslash [HW(3)]SU(2) \simeq U(1) \backslash HW(3)$ , with intrinsic spin degrees of freedom defined by a choice of intrinsic symmetry group  $H = U(2)$ .

In the following sections, we show that VCS theory produces three categories of quantization of an algebraic model with intrinsic degrees of freedom: (i) semi-classical partial quantizations for which only the intrinsic degrees of freedom are quantized; (ii) unitary reducible representations that have the form of a prequantization; and (iii) unitary irreps of a full quantization, equivalent to those obtained by GQ but with an additional fibre structure encompassing the intrinsic degrees of freedom. Each category of representation is a natural extension of the scalar theory.

### 3. Semi-classical partial quantizations

A *partial quantization* is a representation in which only the intrinsic degrees of freedom are quantized and the extrinsic degrees of freedom are represented classically.

To be specific, suppose that  $M$  is an irreducible unitary representation of an intrinsic symmetry group  $H$  on a finite-dimensional (intrinsic) Hilbert space  $U$ . Then a partial quantization is obtained by replacing the classical phase space, seen as a  $H_\rho \backslash G \rightarrow H \backslash G$  bundle with typical fibre  $H_\rho \backslash H$ , by a semi-classical state space  $B$  with the geometric structure of a fibre bundle associated with the principal  $G \rightarrow H \backslash G$  bundle by the representation  $M$

of  $H$ . A semi-classical state of the system, corresponding to a point in  $B$ , is then a state vector in the intrinsic Hilbert space over a point in the classical  $H \backslash G$  phase space.

Semi-classical representations result when the scalar coherent state construction of a classical representation is generalized to a VCS construction. As a prelude to defining  $H$  and  $M$ , we start with a finite-dimensional subspace  $U \subset \mathbb{H}$  of the Hilbert space for an abstract unitary representation  $T$  of the dynamical group  $G$ . Denote by  $E$  the natural embedding  $E : U \rightarrow \mathbb{H}$ . There is then a system  $\{U(g); g \in G\}$  of coherent state subspaces in  $\mathbb{H}$  defined by

$$U(g) = \{|\psi(g)\rangle = T(g^{-1})|\psi\rangle; |\psi\rangle \in E(U)\}. \quad (8)$$

Let  $\Pi$  denote the projection of  $\mathbb{H}$  to  $U$  relative to the inner product on  $\mathbb{H}$ . Then the subspace  $U \subset \mathbb{H}$  defines a map  $\hat{\rho} : \mathfrak{g} \rightarrow GL(U)$  from the Lie algebra  $\mathfrak{g}$  to the linear transformations of  $U$  by

$$\hat{\rho}(A) = \Pi \hat{A} E \quad \forall A \in \mathfrak{g}. \quad (9)$$

In the special case that  $U$  is one-dimensional and spanned by a state of unit norm  $|0\rangle$ ,  $\hat{\rho}$  reduces to a scalar density and acts on an arbitrary vector  $|\psi\rangle \in U$  by scalar multiplication, i.e.  $\hat{\rho}(A)|\psi\rangle = |\psi\rangle\langle 0|\hat{A}|0\rangle$ . Thus, the above definition of  $\hat{\rho}$  generalizes the concept of a density  $\rho : \mathfrak{g} \rightarrow \mathbb{R}$  to a map  $\hat{\rho} : \mathfrak{g} \rightarrow GL(U)$ ; we therefore refer to  $\hat{\rho}$  as a *semi-classical density*. The set of such semi-classical densities

$$\mathcal{O}_{\hat{\rho}} = \{\hat{\rho}_g; g \in G\} \quad (10)$$

defined by

$$\hat{\rho}_g(A) = \hat{\rho}(A(g)) \quad (11)$$

is then a natural generalization of a coadjoint orbit.

The orbit  $\mathcal{O}_{\hat{\rho}}$  has the structure of a fibre bundle over  $H \backslash G$ , where  $H$  is a subgroup of  $G$  with Lie algebra

$$\mathfrak{h} = \{A \in \mathfrak{g} | \hat{\rho}([A, X]) = [\hat{\rho}(A), \hat{\rho}(X)], \forall X \in \mathfrak{g}\}. \quad (12)$$

With this definition,  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{g}$  for which the restriction of  $\hat{\rho}$  to  $\mathfrak{h} \subset \mathfrak{g}$  is a representation. Let  $M$  be an extension of this representation to the group  $H$  such that

$$i \frac{d}{dt} M(e^{-iAt}) \Big|_{t=0} = M(A) \equiv \hat{\rho}(A) \quad \forall A \in \mathfrak{h} \quad (13)$$

and

$$\hat{\rho}(X(hg)) = M(h)\hat{\rho}(X(g))M(h^{-1}) \quad \forall h \in H \quad X \in \mathfrak{g}. \quad (14)$$

The elements of  $\mathcal{O}_{\hat{\rho}}$  then satisfy the  $H$ -equivariance condition

$$\hat{\rho}_{hg} = M(h)\hat{\rho}_g M(h^{-1}) \quad \forall h \in H \quad (15)$$

and  $\mathcal{O}_{\hat{\rho}}$  is interpreted as a fibre bundle over  $H \backslash G$  associated with the principal  $G \rightarrow H \backslash G$  bundle by the action (15). The  $H$ -equivariance condition is a generalization of the  $H$ -invariance condition for the scalar densities of a standard coadjoint orbit  $\mathcal{O}_{\rho} \sim H_{\rho} \backslash G$ ;

$$\rho_{hg} = \chi(h)\rho_g\chi(h^{-1}) = \rho_g \quad \forall h \in H_{\rho}. \quad (16)$$

It is interesting to note that the representation  $M$  of  $\mathfrak{h}$  and  $H$ , defined by

$$M : A \rightarrow \hat{\rho}(A) \quad \forall A \in \mathfrak{h} \quad (17)$$

is generally not a subrepresentation of the restriction of the representation  $T$  to  $\mathfrak{h} \subset \mathfrak{g}$ . The parallel of this observation was obvious for the Abelian scalar representation

$$\chi : A \rightarrow \langle 0|T(A)|0\rangle \quad A \in \mathfrak{h}_{\rho} \quad (18)$$

but it is less obvious that multidimensional representations that are not subrepresentations exist. However, they are known for some Lie algebras and are described as *embedded* representations [8].

Note also that the representation  $M$  could be reducible. However, although it is not essential, we shall assume in the following that the subspace  $U \subset \mathbb{H}$  is chosen in such a way that it is irreducible.

The semi-classical density now defines a semi-classical representation of  $\mathfrak{g}$  in which an element  $A \in \mathfrak{g}$  is mapped to an operator-valued function  $\hat{A}$  over  $G$  having values

$$\hat{A}(g) = \hat{\rho}_g(A) = \hat{\rho}(A(g)) \tag{19}$$

in  $GL(U)$ , which satisfies the equivariance relationship

$$\hat{A}(hg) = M(h)\hat{A}(g)M(h^{-1}) \quad \forall h \in H. \tag{20}$$

The Poisson bracket for this representation is defined by

$$\{\hat{A}, \hat{B}\}(g) = -\frac{i}{\hbar}\hat{\rho}([A(g), B(g)]). \tag{21}$$

Let  $\{A_i\}$  be a basis for  $\mathfrak{h}$  and  $\{A_\nu\}$  a complementary set that completes a basis for  $\mathfrak{g}$ . From the expansion

$$A(g) = \sum_i A^i(g)A_i + \sum_\nu A^\nu(g)A_\nu \tag{22}$$

it follows that

$$\hat{A}(g) = \sum_i A^i(g)M(A_i) + \sum_\nu A^\nu(g)\hat{\rho}(A_\nu) \tag{23}$$

and that

$$\{\hat{A}, \hat{B}\}(g) = -\frac{i}{\hbar}[\hat{A}(g), \hat{B}(g)] + \sum_{\mu\nu} A^\mu(g)\hat{\Omega}_{\mu\nu}B^\nu(g) \tag{24}$$

where

$$\hat{\Omega}_{\mu\nu} = -\frac{i}{\hbar}(\hat{\rho}([A_\mu, A_\nu]) - [\hat{\rho}(A_\mu), \hat{\rho}(A_\nu)]). \tag{25}$$

Following standard terminology, it is convenient to characterize the decomposition of a Lie algebra element into a vertical component (an element of  $\mathfrak{h}$ ) and a complementary (horizontal) component, as a choice of gauge. Thus, a gauge is defined by a projection  $\mathfrak{g} \rightarrow \mathfrak{h}; A(g) \mapsto \sum_i A^i(g)A_i$ . It is then notable that the second term of equation (24) is gauge independent. This independence follows from the definition of  $\mathfrak{h}$ , equation (12), which implies that

$$\sum_{\mu\nu} A^\mu(g)\hat{\Omega}_{\mu\nu}B^\nu(g) = -\frac{i}{\hbar}(\hat{\rho}([A(g), B(g)]) - [\hat{\rho}(A(g)), \hat{\rho}(B(g))]). \tag{26}$$

Consequently, as shown in the appendix, the semi-classical Poisson bracket of equation (24) has a manifestly covariant expression

$$i\hbar\{\hat{A}, \hat{B}\}(g) = [\hat{A}(g), \hat{B}(g)] + i\hbar\hat{\Omega}(X_{\hat{A}}(g), X_{\hat{B}}(g)) \tag{27}$$

where  $X_{\hat{A}}$  is a Hamiltonian vector field generated by  $\hat{A}$  and  $\hat{\Omega}$  is a curvature tensor for the semi-classical phase space (both of which are defined in the appendix).

While for formal purposes it is convenient to express a classical representation by functions over the group  $G$ , it is generally more useful, in practical applications, to represent them as functions over a suitable set of  $H \backslash G$  coset representatives. Recall that a set of coset

representatives  $K = \{k(g) \in Hg; g \in G\}$  defines a factorization  $g = h(g)k(g)$ , with  $h(g) \in H$ , of every  $g \in G$ . Hence, it follows from the identity

$$\hat{A}(h(g)k(g)) = M(h(g))\hat{A}(k(g))M(h^{-1}(g)) \quad (28)$$

that, given the representation  $M$ , the restriction of  $\hat{A}$  to the subset  $K \subset G$  is sufficient to uniquely define  $\hat{A}$ . Moreover, the Poisson bracket of two such functions is given directly in terms of this restriction by

$$\{\hat{A}, \hat{B}\}(k) = -\frac{i}{\hbar}\hat{\rho}([A(k), B(k)]) \quad \forall k \in K. \quad (29)$$

Often it is convenient to consider factorizations of the type  $g = h(g)k(g)$  with  $h(g) \in H^c$  and  $k(g) \in K$ , where  $K$  is a subset of  $H^c \backslash G^c$  coset representatives and  $H^c$  and  $G^c$  are the complex extensions of  $H$  and  $G$ , respectively. The semi-classical representation is then by operator-valued functions on  $K$ .

As an illustration of partial quantization, suppose the intrinsic spin observables of a particle in a three-dimensional Euclidean space, cf section 2, are described quantally by a finite-dimensional irrep  $M$  of the  $u(2)$  intrinsic symmetry algebra. Let  $\{\xi_{sm}; m = -s, \dots, s\}$  be an orthonormal basis for the Hilbert space  $U$  of this irrep. Let  $E : U \rightarrow \mathbb{H}; \xi_{sm} \mapsto |sm\rangle$  be an embedding of  $U$  as an  $su(2)$ -invariant subspace of  $\mathbb{H}$  such that

$$\langle sm|\hat{q}_i|sn\rangle = \langle sm|\hat{p}_i|sn\rangle = 0 \quad \langle sm|\hat{I}|sn\rangle = \delta_{mn} \quad (30)$$

and define

$$\hat{\rho}(A) = \sum_{mn} \xi_{sm} \langle sm|\hat{A}|sn\rangle \xi_{sn}^\dagger \quad \forall A \in \mathfrak{g} \quad (31)$$

with the understanding that  $\xi_{sn}^\dagger \cdot \xi_{sm} = \delta_{mn}$ . The semi-classical representation of the  $[hw(3)]su(2)$  algebra can be defined on the coset space  $U(2) \backslash [HW(3)]SU(2)$  (i.e. the  $p$ - $q$  plane) as

$$\begin{aligned} \hat{Q}_i(p, q) &= \hat{\rho}(\hat{q}_i(g)) = q_i \hat{\mathcal{I}} \\ \hat{P}_i(p, q) &= \hat{\rho}(\hat{p}_i(g)) = p_i \hat{\mathcal{I}} \\ \hat{J}_i(p, q) &= \hat{\rho}(\hat{J}_i(g)) = \hat{S}_i + \hat{\mathcal{L}}_i(p, q) \end{aligned} \quad (32)$$

where

$$\hat{S}_i = \hat{\rho}(\hat{J}_i) \quad \hat{\mathcal{L}}_i(p, q) = (q_j p_k - q_k p_j) \hat{\mathcal{I}} \quad (33)$$

are the spin and orbital angular momenta, respectively, and  $\hat{\mathcal{I}} = \hat{\rho}(\hat{I})$  is the identity operator on  $U$ . The quantal part of the Lie bracket for these semi-classical observables is now given by

$$\begin{aligned} [\hat{Q}_i(p, q), \hat{P}_i(p, q)] &= [\hat{J}_i(p, q), \hat{Q}_i(p, q)] = [\hat{J}_i(p, q), \hat{P}_i(p, q)] = 0 \\ [\hat{J}_i(p, q), \hat{J}_j(p, q)] &= i\hbar \hat{S}_k \end{aligned} \quad (34)$$

and the classical part by

$$\begin{aligned} i\hbar \hat{\Omega}(X_{\hat{Q}_i}, X_{\hat{P}_i})(p, q) &= i\hbar \hat{\mathcal{I}} & i\hbar \hat{\Omega}(X_{\hat{J}_i}, X_{\hat{Q}_j})(p, q) &= i\hbar \hat{Q}_k(p, q) \\ i\hbar \hat{\Omega}(X_{\hat{J}_i}, X_{\hat{P}_j})(p, q) &= i\hbar \hat{P}_k(p, q) & i\hbar \hat{\Omega}(X_{\hat{J}_i}, X_{\hat{J}_j})(p, q) &= i\hbar \hat{\mathcal{L}}_k(p, q). \end{aligned} \quad (35)$$

Together, these parts lead to a semi-classical representation of  $[hw(3)]su(2)$  with Poisson bracket given by equation (27).

Such semi-classical representations not only provide a useful and insightful first step in the quantization of a complex system, they are also of considerable physical interest in their

own right. For example, in many situations involving macroscopic degrees of freedom, a classical description of the dynamics is more than adequate. However, macroscopic systems can also have microscopic intrinsic structures for which quantum mechanics is essential. For example, it may be appropriate to quantize the intrinsic dynamics of a heavy molecule but to describe its centre-of-mass motions classically. The scattering of a heavy ion by a nucleus might be another example. The existence of corresponding partial quantizations of their spectrum generating algebras is therefore a potentially powerful tool in their analysis.

#### 4. VCS induced representations as prequantization

A VCS representation can be constructed in the form of a prequantization. It will be convenient to say that an irrep  $M$  of  $H \subset G$  is *contained* in a (possibly projective) representation  $T$  of  $G$  if  $M$  appears in either a direct sum or direct integral decomposition of  $T_H$ , where  $T_H$  is the restriction of  $T$  to  $H \subset G$ . We then say that a semi-classical representation of  $\mathfrak{g}$ , defined by an irrep  $M$  of a compact intrinsic symmetry group  $H \subset G$ , is quantizable if  $M$  is contained in some unitary representation  $T$  of the group  $G$  on a Hilbert space  $\mathbb{H}$ . It follows, by Schur's lemma, that if  $M$  is quantizable there exists a non-vanishing  $H$ -intertwining operator  $\Pi : \mathbb{H}_D \rightarrow U$ , from a dense subspace of  $\mathbb{H}$  to  $U$ , the carrier space of  $M$ , such that

$$\Pi T(h) = M(h)\Pi \quad \forall h \in H. \quad (36)$$

Given an abstract unitary representation  $T$  of  $G$  and such an  $H$ -intertwining operator, a VCS wavefunction  $\Psi$  is defined over  $G$  [7] for every  $|\Psi\rangle \in \mathbb{H}_D$  by

$$\Psi(g) = \Pi T(g)|\Psi\rangle \quad \forall g \in G. \quad (37)$$

It follows from the definition of  $\Pi$  that

$$\Psi(hg) = M(h)\Psi(g) \quad \forall h \in H. \quad (38)$$

A VCS representation  $\Gamma$  of the group  $G$  induced from the representation  $M$  of the subgroup  $H \subset G$ , is then defined by

$$[\Gamma(g')\Psi](g) = \Psi(gg') \quad g' \in G. \quad (39)$$

Equations (38) and (39), of which the scalar coherent state representations are special cases, are the basic equations of all inducing constructions.

For example, suppose  $M$  is a representation of  $H$  on a Hilbert space  $U$  with orthonormal basis  $\{\xi_m\}$  and  $E : U \rightarrow \mathbb{H}$ ;  $\xi_m \mapsto |m\rangle$  is an embedding of  $U$  as an  $H$ -invariant subspace  $E(U)$  in  $\mathbb{H}$ . Then a suitable intertwining operator is defined by

$$\Pi = \sum_m \xi_m \langle m| \quad (40)$$

and vector coherent state wavefunctions are expressed as

$$\Psi(g) = \sum_m \xi_m \langle m|T(g)|\Psi\rangle. \quad (41)$$

In principle, the Hilbert space of VCS wavefunctions is determined by the map (37) from  $\mathbb{H}_D$  to VCS wavefunctions; the inner product can be inferred as in section 3.4 of the preceding paper [1]. Many VCS Hilbert spaces are possible depending on the choice of  $T$  and the embedding  $E$ . For example, as discussed briefly in section 6, if  $T$  is the regular representation of the group  $G$  and  $E$  has no special properties, then  $\Gamma$  is the representation of  $G$  induced from



the representation  $M$  of a subgroup  $H \subset G$  in the standard theory of induced representations. This representation is known to be reducible in general and, as we now show, it is a natural generalization of a prequantization. However, the embedding  $E$  can also be chosen such that the VCS representation is a subrepresentation of the standard induced representation. It is shown in the following section that it can even be chosen such that the VCS representation is irreducible.

Following the construction of the scalar coherent state representations, the general inducing construction defines a representation of the Lie algebra  $\mathfrak{g}$  by

$$[\Gamma(A)\Psi](g) = \Pi T(g)T(A)|\Psi\rangle = \Psi(A(g)g) \quad A \in \mathfrak{g} \quad (42)$$

where  $\Psi(Ag)$  is defined generally, for any  $A \in \mathfrak{g}$ , by

$$\Psi(Ag) = i \frac{d}{dt} \Psi(e^{-itA}g) \Big|_{t=0}. \quad (43)$$

For a given choice of gauge, defined by a basis  $\{A_i\}$  for  $\mathfrak{h}$  and a complementary set  $\{A_v\}$  to complete a basis for  $\mathfrak{g}$ , the expansion of  $A(g)$  given by equation (22) leads to the explicit expression

$$[\Gamma(A)\Psi](g) = \sum_i A^i(g)M(A_i)\Psi(g) + i\hbar \sum_v A^v(g)[\partial_v\Psi](g) \quad (44)$$

where

$$[\partial_v\Psi](g) = \frac{\partial}{\partial x^v} \Psi \left( \exp \left( -\frac{i}{\hbar} \sum_\mu x^\mu A_\mu \right) g \right) \Big|_{x=0}. \quad (45)$$

Note that this generalization of a scalar coherent state representation is achieved simply by replacing the one-dimensional representation  $\chi$  of the intrinsic symmetry group by the multidimensional representation  $M$ .

Like its scalar counterpart, the representation  $\Gamma$  can be expressed in the covariant form of a prequantization. From equation (23), we have

$$\sum_i A^i(g)M(A_i) = \hat{A}(g) - \sum_v A^v(g)\hat{\rho}(A_v). \quad (46)$$

Equation (44) then becomes

$$[\Gamma(A)\Psi](g) = \hat{A}(g)\Psi(g) + i\hbar \sum_v A^v(g)[\nabla_v\Psi](g) \quad (47)$$

where

$$\nabla_v = \partial_v + \frac{i}{\hbar} \hat{\rho}(A_v). \quad (48)$$

The first term,  $\hat{A}(g)\Psi(g)$ , of equation (47) is manifestly covariant. Moreover, from definition (42), the second term is identical to

$$i\hbar[\nabla_A\Psi](g) = \Psi(A(g)g) - \hat{\rho}(A(g))\Psi(g) \quad (49)$$

where

$$[\nabla_A\Psi](g) = \sum_v A^v(g)[\nabla_v\Psi](g). \quad (50)$$

Thus, it too is covariant.

It is shown in the appendix that  $\nabla_A$  is identical to the covariant derivative  $\nabla_{X_{\hat{A}}}$  in the direction of the vector field  $X_{\hat{A}}$  and is expressed in a particular gauge as a sum

$$\nabla_A = \nabla_{X_{\hat{A}}} = X_{\hat{A}} + \frac{i}{\hbar} \hat{\theta}(X_{\hat{A}}) \quad (51)$$

where  $X_{\hat{A}}$  is a Hamiltonian vector field generated by  $\hat{A}$  and  $\hat{\theta}$  is a one-form. It is also shown that the curvature  $\hat{\Omega}$  of the semi-classical phase space is the covariant exterior derivative of  $\hat{\theta}$  given by

$$\hat{\Omega}(X_{\hat{A}}, X_{\hat{B}}) = d\hat{\theta}(X_{\hat{A}}, X_{\hat{B}}) - [\hat{\theta}(X_{\hat{A}}), \hat{\theta}(X_{\hat{B}})]. \tag{52}$$

Thus, the VCS representation  $\Gamma(A)$  of an element  $A \in \mathfrak{g}$  is expressed as

$$\Gamma(A) = \hat{A} + i\hbar \nabla_{X_{\hat{A}}} \tag{53}$$

and is seen as a natural generalization of prequantization to include intrinsic degrees of freedom.

As for semi-classical observables, it is generally more useful to express VCS wavefunctions as functions over a suitable set of  $H \backslash G$  coset representatives. Thus, with a set of coset representatives  $K = \{k(g) \in Hg; g \in G\}$ , it follows from the identity

$$\Psi(g) = \Psi(h(g)k(g)) = M(h(g))\Psi(k(g)) \tag{54}$$

that, given  $M$ , the restriction of  $\Psi$  to the subset  $K \subset G$  is sufficient to uniquely define  $\Psi$ . VCS wavefunctions can also be defined over a subset of  $H^c \backslash G^c$  coset representatives  $K^c$  by a factorization  $g = h(g)k(g)$ , with  $h(g) \in H^c, k(g) \in K^c$ , of every  $g \in G$ .

For the example of a particle with intrinsic spin considered in the previous sections, we can take

$$\Pi = \sum_m \xi_{sm} \langle sm | \tag{55}$$

with the previous notations. Then, with

$$\Psi(p, q) = \Pi \exp\left(-\frac{i}{\hbar} \sum_i p_i \hat{q}_i\right) \exp\left(\frac{i}{\hbar} \sum_i q_i \hat{p}_i\right) |\Psi\rangle \tag{56}$$

we obtain the prequantization

$$\begin{aligned} \Gamma(\hat{p}_i) &= -i\hbar \frac{\partial}{\partial q_i} & \Gamma(\hat{q}_i) &= q_i + i\hbar \frac{\partial}{\partial p_i} \\ \Gamma(\hat{J}_i) &= \hat{S}_i - i\hbar \left( p_j \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial p_j} \right) - i\hbar \left( q_j \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial q_j} \right) \end{aligned} \tag{57}$$

which acts on vector-valued functions on  $(p-q)$  space.

### 5. Irreducible representations and quantization

A VCS representation will be irreducible if the intertwining operator  $\Pi$  is such that the only nonzero VCS wavefunctions are those of an irrep. Such irreps are found in VCS theory by a natural generalization of the scalar coherent state construction.

It is known that a representation of an SGA  $\mathfrak{g}$  extends linearly to the complex extension  $\mathfrak{g}^c$  of  $\mathfrak{g}$ . The corresponding extension of a generic unitary representation  $T$  of the real group  $G$  may not converge for all of  $G^c$ . However, it may be sufficient for the purpose of defining an irreducible coherent state representation if the extension of  $T$  is well-defined on  $\mathbb{H}$  for some subset  $U(P) \subset P$  of a subgroup  $P \subset G^c$  which contains  $H$ . Let  $\tilde{M}$  denote an irrep of  $P \subset G^c$  which restricts to a unitary irrep  $M$  of  $H \subset P$ . Now suppose an intertwining operator can be

found such that

$$\psi(zg) = \Pi T(z)T(g)|\psi\rangle = \tilde{M}(z)\psi(g) \quad \forall z \in U(P). \quad (58)$$

We then say that the irrep  $\Gamma$  is induced from the representation  $\tilde{M}$  of  $P$ . It will be shown by examples in the following sections that, for many categories of groups, there are natural choices of  $P$  and its representation  $\tilde{M}$  for which the corresponding VCS representation is irreducible.

Subgroups which lead to irreducible induced representations are familiar in representation theory. For example, if  $G$  were semisimple and the intrinsic symmetry group  $H$  were a Levi subgroup, a suitable subgroup  $P \subset G^c$  would be the parabolic subgroup generated by  $H$  and the exponentials of a set of raising (or lowering) operators.

Apart from imposing the stronger condition (58), the coherent state construction is the same as in section 4. However, the stronger condition restricts the set of coherent state wavefunctions to a subset with the result that the coherent state representation becomes an irreducible subrepresentation of that given in section 4.

Now if a unitary coherent state representation  $\Gamma$  of a dynamical group  $G$  induced from a representation  $\tilde{M}$  of a subgroup  $P \subset G^c$  defines an irreducible representation of the Lie algebra  $\mathfrak{g}$  and if the representation  $\tilde{M}$  satisfies the equality

$$i \frac{d}{dt} \tilde{M}(e^{-iAt}) \Big|_{t=0} = \tilde{M}(A) \equiv \hat{\rho}(A) \quad A \in \mathfrak{p} \quad (59)$$

then we say that  $\Gamma$  is a quantization of the classical representation of  $\mathfrak{g}$  defined by  $\hat{\rho}$ .

Note, however, that for this quantization condition to be satisfied, the classical representation corresponding to the density  $\hat{\rho}$  must define a representation  $\tilde{M}$  of a subalgebra  $\mathfrak{p} \subset \mathfrak{g}^c$  that is contained in a unique irrep of  $\mathfrak{g}^c$  which restricts to a unitary irrep of  $\mathfrak{g}$ . This irrep of  $\mathfrak{g}$  must integrate to a (possibly projective) irrep of  $G$ .

The above VCS quantization of a classical model is a practical expression of induced representation theory in the language of geometric quantization. Evidently the subgroup  $P \subset G^c$  used to construct an irreducible VCS induced representation defines an invariant polarization of the tangent space at each point of the base manifold  $H \backslash G$  of the semi-classical bundle provided its Lie algebra  $\mathfrak{p}$  satisfies the conditions:

- (i)  $\hat{\rho}([A, B]) = [\hat{\rho}(A), \hat{\rho}(B)]$  for any  $A, B \in \mathfrak{p}$ ,
- (ii)  $\dim_{\mathbb{R}} \mathfrak{g} + \dim_{\mathbb{R}} \mathfrak{h} = 2 \dim_{\mathbb{C}} \mathfrak{p}$ ,
- (iii)  $\mathfrak{p}$  is invariant under the adjoint action of  $H$ .

The first condition ensures that the polarization is isotropic in the sense that  $\hat{\Omega}(A, B) = 0$  for all  $A, B \in \mathfrak{p}$ . The second condition ensures that  $\mathfrak{p}$  is a maximal subalgebra for which the first condition holds. The final condition ensures that the polarization is well-defined on  $H \backslash G$ . In all the examples we consider, these conditions are satisfied by the Lie algebra  $\mathfrak{p} \subset \mathfrak{g}^c$  used in the VCS construction.

For the example of a particle with intrinsic spin considered in the previous sections, we can take as a polarization the subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}^c$  spanned by the elements  $\{\hat{I}, \hat{J}_i, \hat{q}_i\}$ . Let  $\tilde{M}$  denote the representation of  $\mathfrak{p}$  which restricts to the previous representation  $M$  of  $u(2)$  and to the zero representation of the Abelian algebra spanned by  $\{\hat{q}_i\}$ ; i.e.,  $\tilde{M}(\hat{q}_i) = 0$ . Then, with  $\Pi = \sum_m \xi_{sm} \langle sm |$  defined such that

$$\begin{aligned} \sum_m \xi_{sm} \langle sm | \hat{J}_i | \Psi \rangle &= \hat{S}_i \sum_m \xi_{sm} \langle sm | \Psi \rangle \\ \sum_m \xi_{sm} \langle sm | \hat{q}_i | \Psi \rangle &= 0 \end{aligned} \quad (60)$$

so that  $\langle sm|$  is a functional on a dense subspace of  $\mathbb{H}$ , we obtain  $p_i$ -independent VCS wavefunctions and the irreducible representation

$$\begin{aligned} \Gamma(\hat{p}_i) &= -i\hbar \frac{\partial}{\partial q_i} & \Gamma(\hat{q}_i) &= q_i \\ \Gamma(\hat{J}_i) &= \hat{S}_i - i\hbar \left( q_j \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial q_j} \right) \end{aligned} \tag{61}$$

of a full quantization.

**6. VCS inner products and Hilbert spaces**

Let  $U$  denote a Hilbert space with orthonormal basis  $\{\xi_\nu\}$  for a finite-dimensional unitary irrep  $M$  of a subgroup  $H \subset G$ .

We consider first the situation in which  $U$  can be identified with an  $H$ -invariant subspace of the Hilbert space  $\mathbb{H}$  for some unitary representation  $T$  by an embedding  $E : U \rightarrow \mathbb{H}; \xi_\nu \mapsto |\nu\rangle$ . The corresponding  $\mathbb{H} \rightarrow U$  projection operator

$$\Pi = \sum_\nu \xi_\nu \langle \nu| \tag{62}$$

then satisfies the equation

$$M(h)\Pi = \Pi T(h) \quad \forall h \in H. \tag{63}$$

Thus,  $\Pi$  is an  $H$ -intertwining operator and defines a set of VCS wavefunctions

$$\Psi(g) = \Pi T(g)|\Psi\rangle = \sum_\nu \xi_\nu \Psi_\nu(g) \quad g \in G \quad |\Psi\rangle \in \mathbb{H}. \tag{64}$$

Now, if  $U$  is contained in a subrepresentation of  $T$  which is a direct sum of discrete series representations, the operator

$$\mathbb{I} = \int_G \sum_\nu T(g^{-1})|\nu\rangle \langle \nu| T(g) d\nu(g) \tag{65}$$

where  $d\nu$  is a left-invariant measure on  $G$ , is well-defined on  $\mathbb{H}$ . Moreover it commutes with the representation  $T(g)$  of any element  $g \in G$ . Thus, by Schur's lemma,  $\mathbb{I}$  acts as a multiple of the identity on any irreducible subspace of  $\mathbb{H}$ . Thus, an inner product is defined for the VCS wavefunctions by

$$\begin{aligned} (\Psi, \Psi') &= \langle \Psi | \mathbb{I} | \Psi' \rangle = \int_G \Psi^*(g) \cdot \Psi'(g) d\nu(g) \\ &= \int_G \sum_\nu \Psi_\nu^*(g) \Psi'_\nu(g) d\nu(g). \end{aligned} \tag{66}$$

However, because  $\Psi(hg) = M(h)\Psi(g)$  for  $h \in H$ , the scalar product in  $U$  satisfies

$$\Psi^*(hg) \cdot \Psi'(hg) = \Psi^*(g) \cdot \Psi'(g) \tag{67}$$

and the integral over  $G$  in equation (66) can be restricted to an integral over the coset space  $H \backslash G$  with respect to the left  $H$ -invariant measure inherited from  $G$ .

The above construction works when  $M$  is a subrepresentation of the restriction of  $T$  to  $H \subset G$ . If  $M$  is not a subrepresentation but is contained in a direct integral decomposition of the restriction of  $T$  to  $H$ , then it is still possible to define an  $H$ -intertwining operator by equation (62) that satisfies equation (63) albeit with  $\{\langle \nu| \}$  defined as a set of functionals on a dense subspace  $\mathbb{H}_D$  of  $\mathbb{H}$ . It can then happen that the integral expression for  $\mathbb{I}$  may not converge.

However, the corresponding integral over  $H \backslash G$  may converge and, if so, it defines an inner product for VCS wavefunctions in parallel with Mackey's construction of inner products for induced representations. Inner products for more general VCS representations are constructed by  $K$ -matrix methods [9].

The Hilbert space of all VCS wavefunctions that satisfy the constraint equation (38) and are normalizable with respect to the above-defined inner product is that of the standard representation of  $G$  induced from the representation  $M$  of the subgroup  $H \subset G$ . The subspace of VCS wavefunctions that satisfy the stronger constraint condition (58) for a suitable polarization is the Hilbert space for an irreducible induced representation.

## 7. Examples of VCS representations

The  $SU(3)$  and rigid rotor models provide insightful and representative examples of the VCS quantization methods. Despite its apparent simplicity, the quantization of rotational models is considerably more difficult than traditional canonical problems with three degrees of freedom. The difficulties arise from the nontrivial geometry of the phase spaces and the possibility of intrinsic degrees of freedom. However, the VCS quantization techniques handle these problems with ease. In the following, algebraic formulations of both the  $SU(3)$  and rotor models will be given, and the techniques of the previous sections will be used to investigate their classical, semi-classical and quantal realizations with intrinsic degrees of freedom.

### 7.1. Coherent state representations of $SU(3)$

An  $su(3)$  model was first formulated as an algebraic model of nuclear rotations by Elliott [10]. It was followed by the  $su(3)$  quark model of Gell-Mann and Ne'eman [11]. These models have enjoyed enormous successes partly because of their simplicity; the  $su(3)$  algebra is semi-simple and has a straightforward and well understood representation theory; it is also compact and its unitary irreps are finite dimensional. VCS theory was applied to  $su(3)$  in [12] and reviewed in [13, 14].

Let  $\{C_{ij}; i, j = 1, 2, 3\}$  be the standard basis for  $gl(3, \mathbb{C}) \simeq u(3)^c$  with commutation relations

$$[C_{ij}, C_{kl}] = \delta_{jk}C_{il} - \delta_{il}C_{kj}. \quad (68)$$

Then  $su(3)$  is the real linear span of the Hermitian combinations

$$\begin{aligned} J_{ij} &= -i(C_{ij} - C_{ji}) & i < j \\ Q_{ij} &= (C_{ij} + C_{ji}) & i < j \\ H_i &= (C_{ii} - C_{i+1, i+1}) & 1 \leq i \leq 2. \end{aligned} \quad (69)$$

Let  $T$  denote the regular representation of the group  $SU(3)$ . It can be extended to a representation of  $SL(3, \mathbb{C})$  on the algebraic direct sum of the irreps of  $SU(3)$ , which is dense in the regular representation. As usual we denote by  $A \rightarrow \hat{A} = T(A)$  the corresponding representation of the Lie algebra  $sl(3, \mathbb{C})$ . The coherent state methods outlined lead to several classes of  $su(3)$  representations corresponding to: classical representations, semi-classical representations of a partial quantization, the induced representations of prequantization and the irreducible unitary representations of a full quantization.

*7.1.1. Classical representations.* Scalar coherent state techniques lead to a classical representation as follows. Let  $|0\rangle$  be some state in the Hilbert space  $\mathbb{H}$  of the representation  $T$

for which

$$\begin{aligned} \langle 0|\hat{C}_{ij}|0\rangle &= 0 & i \neq j \\ \langle 0|\hat{H}_i|0\rangle &= v_i. \end{aligned} \tag{70}$$

By the standard moment map, a classical density  $\rho \in su(3)^*$  is defined by

$$\rho(X) = \langle 0|\hat{X}|0\rangle \quad X \in su(3) \tag{71}$$

and extended linearly to elements of  $su(3)^c$  in the usual way by setting  $\rho(X + iY) = \rho(X) + i\rho(Y)$ . A classical phase space is defined as the coadjoint orbit  $\mathcal{O}_\rho = \{\rho_g; g \in SU(3)\}$ , where

$$\rho_g(A) = \rho(A(g)) = \langle 0|T(g)\hat{A}T(g^{-1})|0\rangle \quad A \in su(3). \tag{72}$$

This phase space is diffeomorphic to the factor space  $H_\rho \backslash SU(3)$ , where  $H_\rho$  is the isotropy subgroup

$$H_\rho = \{h \in SU(3) \mid \rho_h = \rho\}. \tag{73}$$

We consider the generic situation, in which  $H_\rho$  is the Cartan subgroup with Lie algebra spanned by  $H_1$  and  $H_2$ . (When  $v_1$  or  $v_2$  is zero, for example,  $H_\rho$  is a larger subgroup and the construction simplifies.) A classical representation of  $su(3)$  is then defined in which an element  $A \in su(3)^c$  maps to a function  $\mathcal{A}$  on  $H_\rho \backslash SU(3)$  with values

$$\mathcal{A}(g) = \rho_g(A) = \rho(A(g)). \tag{74}$$

The Poisson bracket for this classical representation is defined in the standard way by

$$\{\mathcal{A}, \mathcal{B}\}(g) = \omega_g(A, B) = -\frac{i}{\hbar} \rho_g([A, B]) \tag{75}$$

for  $A, B \in su(3)$ .

The above representation can be obtained in explicit form in terms of suitable coordinate charts for  $H_\rho \backslash SU(3)$  (see examples in [1]). For example, Murnaghan [15] has shown that an  $SU(3)$  matrix can be parameterized by the factorization

$$g(\xi, \alpha, \beta) = e^{-i(\xi_1 H_1 + \xi_2 H_2)} g_{23}(\alpha_1, \beta_1) g_{13}(\alpha_2, \beta_2) g_{12}(\alpha_3, \beta_3) \tag{76}$$

where

$$g_{23}(\alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -e^{-i\alpha} \sin \beta \\ 0 & e^{i\alpha} \sin \beta & \cos \beta \end{pmatrix} \tag{77}$$

and  $g_{13}$  and  $g_{12}$  are similarly defined. Since the first factor on the rhs of equation (76) is an element of the isotropy subgroup  $H_\rho$ , this parameterization leads to a classical representation of the  $su(3)$  algebra in terms of functions of the  $(\alpha, \beta)$  coordinates.

Now observe that the first two factors on the rhs of equation (76) are elements of a  $U(2) \subset SU(3)$  subgroup. This suggests a fibration of the classical phase space  $H_\rho \backslash SU(3)$  as an intrinsic  $H_\rho \backslash U(2)$  phase space over an extrinsic  $U(2) \backslash SU(3)$  phase space. Because the representation theory of  $U(2)$  is well known, this greatly facilitates the quantization process.

*7.1.2. Semi-classical representations.* The intrinsic symmetry algebra  $u(2)$  suggested by the above parameterization of  $SU(3)$  is spanned by  $H_1$  and the elements of an  $su(2)$  algebra

$$S_z = \frac{1}{2}H_2 \quad S_x = \frac{1}{2}(C_{23} + C_{32}) \quad S_y = -\frac{1}{2}i(C_{23} - C_{32}). \tag{78}$$

Thus, for the intrinsic degrees of freedom of the  $SU(3)$  classical phase space to be quantizable, it is required that  $v_2$  should be an integer. Moreover, in order that it should be an  $su(2)$  highest

weight and uniquely define an  $su(2)$  irrep, it should be a positive integer. The representation label  $\nu_1$  is not so constrained. For, if it is not an integer, the only consequence is that the associated representation of the  $u(1)$  algebra integrates to a unitary projective representation of  $U(1)$ , i.e. a unitary representations of a covering group of  $U(1)$ . This is not possible for  $su(2)$  because the group  $SU(2)$  is simply connected; it is its own universal covering group.

Let  $M$  denote a unitary (possibly projective) irrep of the  $U(2)$  intrinsic symmetry group of highest weight  $(\nu_1, \mu)$  (with  $\mu$  a positive integer) on a Hilbert space  $U$ . Let  $E : U \rightarrow \mathbb{H}$  be an embedding of  $U$  in the regular representation  $\mathbb{H}$  and let  $\Pi : \mathbb{H} \rightarrow U$  be the corresponding orthogonal projection with respect to the inner product for  $\mathbb{H}$ . The embedding  $E$  is required to be such that

$$\Pi \hat{A} E = M(A) \quad \forall A \in u(2) \subset su(3). \quad (79)$$

Now define

$$\hat{\rho}(A) = \Pi \hat{A} E \quad A \in su(3) \quad (80)$$

and assume, for convenience, that  $E$  is chosen such that

$$\hat{\rho}(C_{12}) = \hat{\rho}(C_{13}) = \hat{\rho}(C_{21}) = \hat{\rho}(C_{31}) = 0. \quad (81)$$

Then

$$\hat{\rho}(H_1 + \frac{1}{2}H_2) = \nu_1 + \frac{1}{2}\mu \quad \hat{\rho}(S_i) = \hat{S}_i \quad (82)$$

where we draw attention to the fact  $(H_1 + \frac{1}{2}H_2)$  commutes with the  $su(2)$  operators  $\{S_i\}$  but  $H_1$  on its own does not.

A partial quantization of  $su(3)$  is now defined as a semi-classical representation in which an element  $A \in su(3)$  is mapped to a  $U(2)$ -equivariant operator-valued function  $\hat{A}$  on  $SU(3)$  with values

$$\hat{A}(g) = \hat{\rho}(A(g)). \quad (83)$$

Note that, because

$$\hat{A}(hg) = M(h)\hat{A}(g)M(h^{-1}) \quad \forall h \in U(2) \quad (84)$$

it is sufficient to evaluate the classical operator-valued functions and their Poisson brackets on a set of  $U(2) \backslash SU(3)$  coset representatives (cf section 3). Thus, making use of the Murnaghan factorization of equation (76), a semi-classical representation is defined over a set of  $U(2) \backslash SU(3)$  coset representatives  $K = \{k(\alpha, \beta)\}$  with

$$k(\alpha, \beta) = g_{13}(\alpha_2, \beta_2)g_{12}(\alpha_3, \beta_3) \quad (85)$$

for a suitable range of  $(\alpha, \beta)$  values.

The expressions for this semi-classical representation as operator-valued functions of  $(\alpha, \beta)$  can be worked out. However, they are expressed more simply in terms of coset representatives

$$K = \{e^{Y(y)} e^{Z(z)}\} \quad (86)$$

for  $H^c \backslash G^c$  for which  $Y(y)$  and  $Z(z)$  are linear combinations of commuting Lie algebra elements, i.e.

$$Y(y) = y_2 C_{21} + y_3 C_{31} \quad Z(z) = z_2 C_{12} + z_3 C_{13}. \quad (87)$$

From the identities

$$\begin{aligned} e^{Y(y)} e^{Z(z)} C_{12} e^{-Z(z)} e^{-Y(y)} &= C_{12} - y_2 H_1 + y_3 C_{32} - y_2^2 C_{21} - y_2 y_3 C_{31} \\ e^{Y(y)} e^{Z(z)} C_{13} e^{-Z(z)} e^{-Y(y)} &= C_{13} - y_3 (H_1 + H_2) + y_2 C_{23} - y_2 y_3 C_{21} - y_3^2 C_{31} \\ e^{Y(y)} e^{Z(z)} C_{23} e^{-Z(z)} e^{-Y(y)} &= (1 + y_2 z_2) C_{23} - y_3 z_2 (H_1 + H_2) \\ &\quad - y_3 (1 + y_2 z_2) C_{21} + z_2 C_{13} - z_2 y_3^2 C_{31} \end{aligned} \quad (88)$$

it follows that the semi-classical representations of the elements  $C_{12}$ ,  $C_{13}$  and  $C_{23}$  are given by

$$\begin{aligned} \hat{C}_{12}(y, z) &= y_3 \hat{S}_- - y_2 \left( v_1 + \frac{1}{2} \mu \right) \hat{I} + y_2 \hat{S}_z \\ \hat{C}_{13}(y, z) &= y_2 \hat{S}_+ - y_3 \left( v_1 + \frac{1}{2} \mu \right) \hat{I} - y_3 \hat{S}_z \\ \hat{C}_{23}(y, z) &= (1 + y_2 z_2) \hat{S}_+ - y_3 z_2 \left( v_1 + \frac{1}{2} \mu \right) \hat{I} - y_3 z_2 \hat{S}_z \end{aligned} \tag{89}$$

where  $\hat{S}_\pm = M(S_x \pm iS_y)$  and  $\hat{I}$  is the unit operator on  $U$ . The representations of all elements of  $su(3)$  can be derived in this fashion. Calculating the semi-classical Poisson bracket  $\{\hat{C}_{12}, \hat{C}_{23}\}(y, z)$  as defined by equation (24), we find that the quantal part is given by

$$[\hat{C}_{12}(y, z), \hat{C}_{23}(y, z)] = y_2(1 + y_2 z_2) \hat{S}_+ - y_3^2 z_2 \hat{S}_- - 2y_3(1 + y_2 z_2) \hat{S}_z \tag{90}$$

and the classical part by

$$i\hbar \hat{\Omega}(X_{\hat{C}_{12}}, X_{\hat{C}_{23}})(y, z) = -y_2^2 z_2 \hat{S}_+ + y_3^2 z_2 \hat{S}_- - y_3 \left( v_1 + \frac{1}{2} \mu \right) \hat{I} + y_3(1 + 2y_2 z_2) \hat{S}_z. \tag{91}$$

Together, these components give

$$\begin{aligned} i\hbar \{\hat{C}_{12}, \hat{C}_{23}\}(y, z) &= [\hat{C}_{12}(y, z), \hat{C}_{23}(y, z)] + i\hbar \hat{\Omega}(X_{\hat{C}_{12}}, X_{\hat{C}_{23}})(y, z) \\ &= y_2 \hat{S}_+ - \left( v_1 + \frac{1}{2} \mu \right) y_3 \hat{I} - y_3 \hat{S}_z \\ &= \hat{C}_{13}(y, z) \end{aligned} \tag{92}$$

as required for a semi-classical representation. However, as we now show, the representations of prequantization and their commutation relations are easier to derive, and those of the irreducible representations of a full quantization are even simpler.

*7.1.3. The induced representations of prequantization.* To be quantizable, the irrep  $M$  of the  $u(2) \subset su(3)$  subalgebra of a semi-classical representation should be a  $u(2)$  irrep contained in some unitary representation  $T$  of  $su(3)$ . This condition requires that  $v_1$  also be a positive integer. Thus, we now suppose that  $M$  is an irrep of  $u(2)$  on an intrinsic Hilbert space  $U$  with highest weight  $(\lambda, \mu)$ , where  $\lambda$  and  $\mu$  are both positive integers. This representation extends to a representation of the  $U(2)$  group.

Let  $T$  be an abstract representation of  $SU(3)$  on a Hilbert space  $\mathbb{H}$  and suppose the irrep  $M$  of  $U(2)$  is contained in  $T$ . Then there exists a  $U(2)$ -intertwining operator  $\Pi : \mathbb{H} \rightarrow U$  satisfying

$$\Pi T(h) = M(h) \Pi \quad \forall h \in U(2). \tag{93}$$

For example, suppose  $V \subset \mathbb{H}$  is an irreducible  $U(2)$ -invariant subspace of  $\mathbb{H}$  with orthonormal basis  $\{|sm\rangle; m = -s, \dots, +s, s = \mu/2\}$  and the intertwining operator

$$\Pi = \sum_m \xi_{sm} \langle sm | \tag{94}$$

maps this basis to a corresponding basis  $\{\xi_{sm}\}$  for  $U$ .

The VCS wavefunctions are now defined over the coset representatives of equation (86);

$$\Psi(y, z) = \Pi e^{\hat{Y}(y)} e^{\hat{Z}(z)} |\Psi\rangle \tag{95}$$

with  $\hat{Y}(y) = y_2 \hat{C}_{21} + y_3 \hat{C}_{31}$  and  $\hat{Z}(z) = z_2 \hat{C}_{12} + z_3 \hat{C}_{13}$ . Thus, for example, the representation  $\Gamma(C_{12})$  of the element  $C_{12} \in su(3)^c$  is given immediately by

$$[\Gamma(C_{12})\Psi](y, z) = \Pi e^{\hat{Y}(y)} e^{\hat{Z}(z)} \hat{C}_{12} |\Psi\rangle = \frac{\partial}{\partial z_2} \Psi(y, z). \tag{96}$$

The representations of other  $su(3)^c$  elements are obtained almost as easily. For example, the expression for one of the most complicated elements, defined by

$$[\Gamma(C_{21})\Psi](y, z) = \Pi e^{\hat{Y}(y)} e^{\hat{Z}(z)} \hat{C}_{21} |\Psi\rangle \tag{97}$$



is obtained from the identities

$$\begin{aligned} e^{\hat{Z}(z)} \hat{C}_{21} &= (\hat{C}_{21} + z_2(\hat{C}_{11} - \hat{C}_{22}) - z_3 \hat{C}_{23} - z_2^2 \hat{C}_{12} - z_2 z_3 \hat{C}_{13}) e^{\hat{Z}(z)} \\ e^{\hat{Y}(y)} (\hat{C}_{11} - \hat{C}_{22}) &= (\hat{C}_{11} - \hat{C}_{22} + 2y_2 \hat{C}_{21} + y_3 \hat{C}_{31}) e^{\hat{Y}(y)} \\ e^{\hat{Y}(y)} \hat{C}_{23} &= (\hat{C}_{23} - y_3 \hat{C}_{21}) e^{\hat{Y}(y)}. \end{aligned} \quad (98)$$

It follows that

$$\begin{aligned} \Gamma(C_{21}) &= (1 + y_3 z_3) \frac{\partial}{\partial y_2} - z_2 \hat{S}_z - z_3 \hat{S} \\ &+ z_2 \left( (\lambda + \frac{1}{2}\mu) + 2y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} - z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} \right). \end{aligned} \quad (99)$$

Similarly, one obtains

$$\Gamma(H_1) = (\lambda + \frac{1}{2}\mu) + 2y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} - 2z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - \hat{S}_z. \quad (100)$$

It is readily checked that these operators satisfy the commutation relations

$$\begin{aligned} [\Gamma(C_{12}), \Gamma(C_{21})] &= \Gamma(H_1) \\ [\Gamma(H_1), \Gamma(C_{12})] &= 2\Gamma(C_{12}) \\ [\Gamma(H_1), \Gamma(C_{21})] &= -2\Gamma(C_{21}). \end{aligned} \quad (101)$$

*7.1.4. The irreducible representations of a full quantization.* For an induced VCS representation of  $su(3)$  to be irreducible, the map  $\Pi : \mathbb{H} \rightarrow U$  must be chosen such that it intertwines a representation of a larger subgroup  $P \subset SU(3)^c$  corresponding to a polarization. Since an irrep of  $SU(3)$  is uniquely defined by its highest weight  $(\lambda, \mu)$ , it is also uniquely defined by an irrep  $\tilde{M}$  of the  $\mathfrak{p} \subset su(3)^c$  subalgebra spanned by the elements  $\{C_{23}, C_{32}, H_1, H_2\}$  of the  $u(2)$  subalgebra, considered for prequantization, together with the operators  $\{C_{21}, C_{31}\}$ . The appropriate irrep is then one for which

$$\tilde{M}(A) = M(A) \quad \forall A \in u(2) \quad (102)$$

and

$$\tilde{M}(C_{21}) = \tilde{M}(C_{31}) = 0. \quad (103)$$

Thus, we take for  $P$  the parabolic subgroup of  $SU(3)^c$  generated by exponentiating the Lie algebra  $\mathfrak{p}$ . The representation  $\tilde{M}$  of  $\mathfrak{p}$  is likewise exponentiated to an irrep of  $P$ . Now if  $\Pi$  is an intertwining operator such that

$$\Pi T(p) = \tilde{M}(p) \Pi \quad \forall p \in P \quad (104)$$

then VCS states are defined by

$$\Psi(z) = \Pi e^{\hat{Z}} |\Psi\rangle \quad (105)$$

with  $\hat{Z}(z) = z_2 \hat{C}_{12} + z_3 \hat{C}_{13}$ . It is immediately seen that such wavefunctions are the  $y$ -independent subset of those of the prequantization of the previous section. Thus, one immediately obtains the operators of an irrep with, for example,

$$\begin{aligned} \Gamma(C_{12}) &= \frac{\partial}{\partial z_2} \\ \Gamma(C_{21}) &= z_2 \left( (\lambda + \frac{1}{2}\mu) - z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} \right) - z_2 \hat{S}_z - z_3 \hat{S}_+ \\ \Gamma(H_1) &= (\lambda + \frac{1}{2}\mu) - 2z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - \hat{S}_z. \end{aligned} \quad (106)$$

This is a standard holomorphic induced representation.

An inner product for this representation is defined such that the representation of the real  $su(3)$  algebra is by Hermitian operators. This inner product leads to an explicit construction of an orthonormal basis for an irrep [16].

7.1.5. *The relationship between VCS and scalar coherent state representations.* A VCS representation can also be expressed as a scalar coherent state representation. However, contrary to what one might expect, the latter is generally more complicated. Consider the above example of a VCS representation of  $SU(3)$ . An equivalent scalar coherent state representation is given by realizing the vectors  $\{\xi_{sm}\}$  in a coherent state representation for  $U(2)$  for which  $\xi_{sm}$ , with  $s = \mu/2$ , becomes a real function of  $SO(2)$ :

$$\xi_{sm}(\theta) = \langle \lambda\mu | e^{i\theta \hat{S}_y} | sm \rangle. \tag{107}$$

A holomorphic VCS wavefunction is then expressed as a scalar coherent state function by observing that

$$\begin{aligned} \Psi(\theta, z) &= \sum_m \xi_{sm}(\theta) \langle sm | e^{\hat{Z}(z)} | \Psi \rangle \\ &= \langle \lambda\mu | e^{i\theta \hat{S}_y} \left( \sum_m | sm \rangle \langle sm | \right) e^{\hat{Z}(z)} | \Psi \rangle \\ &= \langle \lambda\mu | e^{i\theta \hat{S}_y} e^{\hat{Z}(z)} | \Psi \rangle. \end{aligned} \tag{108}$$

The advantage of the VCS representation is that it subsumes all the properties of the chosen subgroup, in this case  $U(2)$ , and thereby avoids having to reproduce them in the expression of the larger group, in this case  $SU(3)$ . However, it is useful to know that a VCS representation can always be expressed as a scalar coherent state representation because it means that any results proved for a scalar CS representation automatically apply, with appropriate interpretation, to a VCS representation.

7.2. *Rigid rotor models*

A classical rigid rotor is characterized by a rigid intrinsic structure. Thus, the dynamical variables of a rigid rotor are its orientation and angular momentum. We consider here an algebraic rotor model with an algebra of observables spanned by the components of the angular momentum and the moments of the inertia tensor for the rotor.

The moments  $\{I_{ij}\}$  of the inertia tensor (in a Cartesian basis) can be viewed as the elements of a real symmetric  $3 \times 3$  matrix. Given values for these observables, the orientation of a rotor is defined (with some ambiguity) by the rotation  $\Omega \in SO(3)$  that brings the inertia tensor to diagonal form,

$$\bar{I}_{ij} = \Omega I \Omega_{ij}^{-1} = \delta_{ij} I_i \tag{109}$$

where  $(I_1, I_2, I_3)$  are fixed intrinsic moments of inertia.

Because the inertia tensor is a function only of orientation, its components commute

$$[I_{ij}, I_{kl}] = 0 \tag{110}$$

and span an algebra isomorphic to  $\mathbb{R}^6$ . The angular momentum  $L$  has Cartesian components  $\{L_i; i = 1, 2, 3\}$  which span an  $so(3)$  Lie algebra,

$$[L_i, L_j] = i\hbar L_k \quad i, j, k \text{ cyclic.} \tag{111}$$

The inertia tensor is defined, by (109), to be a rank-2 Cartesian tensor. Thus, it obeys the commutation relations

$$[I_{ij}, L_k] = i\hbar \sum_l (\varepsilon_{lik} I_{lj} + \varepsilon_{ljk} I_{li}). \quad (112)$$

Together, the moments of inertia and the angular momenta span an SGA for the rotor that is isomorphic to the semidirect sum algebra  $[\mathbb{R}^6]_{SO(3)}$  with  $\mathbb{R}^6$  as its ideal. This algebra is known as the *rotor model algebra* (RMA).

The corresponding dynamical group obtained by exponentiating the RMA is the *rotor model group* (RMG), a group isomorphic to the semidirect product  $[\mathbb{R}^6]SO(3)$ . An element of the RMG is a pair  $(Q, \Omega)$ , with  $Q \in \mathbb{R}^6$  and  $\Omega \in SO(3)$  and the group product is given by

$$(Q_1, \Omega_1) \circ (Q_2, \Omega_2) = (Q_1 + \Omega_1 Q_2 \Omega_1^{-1}, \Omega_1 \Omega_2). \quad (113)$$

This group and its Lie algebra have many classical and quantal representations. The classical representations of rigid rotor models and Euler's equations for their Hamiltonian dynamics are well known. The quantization of the rigid rotor was given by Casimir [17] and is well known in nuclear [18] and molecular physics (cf [19] for a review). The route from classical representations of the rotor to the unitary representations of quantum mechanics is an illuminating example for both the methods of induced representations and of geometric quantization. We show here that the classical and quantal representations have simple expressions in coherent state and VCS theory.

*7.2.1. Classical representations.* A classical representation of a rigid rotor can be derived from any abstract unitary representation  $T$  of the RMA  $[\mathbb{R}^6]_{SO(3)}$  on a Hilbert space  $\mathbb{H}$ . Let  $\hat{A} = T(A)$  for  $A \in [\mathbb{R}^6]_{SO(3)}$ . Let  $|0\rangle$  be a normalized state in  $\mathbb{H}$  and  $\rho_0$  a corresponding density satisfying

$$\rho_0(L_i) = \langle 0 | \hat{L}_i | 0 \rangle = 0 \quad \rho_0(I_{ij}) = \langle 0 | \hat{I}_{ij} | 0 \rangle = \bar{\mathfrak{S}}_{ij} = \delta_{ij} \mathfrak{S}_i \quad (114)$$

with  $i, j = 1, 2, 3$  and  $\mathfrak{S}_i \in \mathbb{R}$ . Then  $\rho_0$  is the element of the dual  $\text{RMA}^*$  that represents a classical state with zero angular momentum and orientation such that the inertia tensor  $\bar{\mathfrak{S}}$  is diagonal, i.e. the principal axes of this inertia tensor coincide with those of the space-fixed coordinate frame. As usual, many classical irreps (in this case with different principal moments of inertia  $\{\mathfrak{S}_i\}$ ) can be derived from a given unitary representation  $T$  by different choices of  $\rho_0$ .

Starting with a density  $\rho_0$ , a classical phase space for the rotor is the coadjoint orbit

$$\mathcal{O}_\rho = \{\rho_{(Q, \Omega)}; (Q, \Omega) \in [\mathbb{R}^6]SO(3)\} \quad (115)$$

of the RMG in  $\text{RMA}^*$ , where  $\rho_{(Q, \Omega)}$  is defined by

$$\begin{aligned} \rho_{(Q, \Omega)}(L_i) &= \langle 0 | T(Q, \Omega) \hat{L}_i T((Q, \Omega)^{-1}) | 0 \rangle \\ \rho_{(Q, \Omega)}(I_{ij}) &= \langle 0 | T(Q, \Omega) \hat{I}_{ij} T((Q, \Omega)^{-1}) | 0 \rangle. \end{aligned} \quad (116)$$

The set of functions  $\{\mathfrak{S}_{ij}, \mathcal{L}_i; i, j = 1, 2, 3\}$ , defined by

$$\begin{aligned} \mathfrak{S}_{ij}(Q, \Omega) &= \rho_{(Q, \Omega)}(I_{ij}) = \sum_k \mathfrak{S}_k \Omega_{ki} \Omega_{kj} \\ \mathcal{L}_l(Q, \Omega) &= \rho_{(Q, \Omega)}(L_l) = -\hbar \sum_{ijk} \varepsilon_{ijk} Q_{ij} (\mathfrak{S}_i - \mathfrak{S}_j) \Omega_{kl} \end{aligned} \quad (117)$$

are then a basis for a classical representation of the RMA with Poisson brackets

$$\begin{aligned} \{\mathfrak{S}_{ij}, \mathfrak{S}_{kl}\}(Q, \Omega) &= -\frac{i}{\hbar} \rho_{(Q, \Omega)}([I_{ij}, I_{kl}]) = 0 \\ \{\mathcal{L}_i, \mathcal{L}_j\}(Q, \Omega) &= -\frac{i}{\hbar} \rho_{(Q, \Omega)}([L_i, L_j]) = \sum_k \varepsilon_{ijk} \mathcal{L}_k(Q, \Omega) \\ \{\mathfrak{S}_{ij}, \mathcal{L}_k\}(Q, \Omega) &= -\frac{i}{\hbar} \rho_{(Q, \Omega)}([I_{ij}, L_k]) = \sum_l (\varepsilon_{lik} \mathfrak{S}_{lj}(Q, \Omega) + \varepsilon_{ljk} \mathfrak{S}_{li}(Q, \Omega)). \end{aligned} \tag{118}$$

If the three principal moments of inertia  $\{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3\}$ , are all different, then the subgroup of rotations that leave the density  $\rho_0$  invariant under the coadjoint action is the discrete group  $D_2$  generated by rotations through angle  $\pi$  about the principal axes and the isotropy subgroup of the phase space is the semidirect product  $[\mathbb{R}^3]D_2$ , where  $\mathbb{R}^3 \subset \mathbb{R}^6$  is the subgroup generated by the diagonal moments  $\{I_{ii}, i = 1, 2, 3\}$ . The phase space  $\mathcal{O}_0 \simeq [\mathbb{R}^3]D_2 \backslash [\mathbb{R}^6]SO(3)$  is then symplectomorphic to the cotangent bundle  $T^*(D_2 \backslash SO(3))$ . This orbit is the phase space of an *asymmetric top*. If two of the principal moments of inertia are equal, e.g.,  $\mathfrak{S}_1 = \mathfrak{S}_2 \neq \mathfrak{S}_3$ , then the subgroup of rotations that leave  $\rho_0$  invariant is  $D_\infty$ , a group comprising rotations about the symmetry axis and rotations through angle  $\pi$  about perpendicular axes. The isotropy subgroup of the phase space is then  $[\mathbb{R}^4]D_\infty$ , where  $\mathbb{R}^4 \subset \mathbb{R}^6$  is the subgroup generated by  $\{I_{ii}, i = 1, 2, 3\}$  and  $I_{12}$ . The phase space  $\mathcal{O}_0 \simeq [\mathbb{R}^4]D_\infty \backslash [\mathbb{R}^6]SO(3)$  is then symplectomorphic to the cotangent bundle  $T^*(D_\infty \backslash SO(3))$  which is the phase space of a *symmetric top*.

The phase space for a symmetric top is of lower dimension than that of an asymmetric top. One of the reasons for this difference is that there is no element of the RMA that can generate a boost in the component of the angular momentum about a symmetry axis. Thus, when  $\mathfrak{S}_1 = \mathfrak{S}_2$ , the component of the angular momentum along the 3-axis is a constant of the motion with value given by that at  $\rho_0$ . This condition does not mean that a symmetric top cannot rotate about its symmetry axis. It means only that it rotates about its symmetry axis with a constant angular momentum. Thus, the component of angular momentum along the symmetry axis of a symmetric top is appropriately regarded as an intrinsic (gauge) degree of freedom.

Consider, for example, a symmetric top representation for which  $\mathfrak{S}_1 = \mathfrak{S}_2 \neq \mathfrak{S}_3$  and, instead of  $|0\rangle$ , consider a normalized state  $|K\rangle$  and corresponding density  $\rho_0^{(K)}$  for which

$$\begin{aligned} \rho_0^{(K)}(L_i) &= \langle K | \hat{L}_i | K \rangle = 0 \quad i = 1, 2 \\ \rho_0^{(K)}(L_3) &= \langle K | \hat{L}_3 | K \rangle = K \\ \rho_0^{(K)}(I_{ij}) &= \langle K | \hat{I}_{ij} | K \rangle = \delta_{ij} \mathfrak{S}_i \end{aligned} \tag{119}$$

where  $K$  is a real constant. The density  $\rho_0^{(K)} \in \text{RMA}^*$  is that of a symmetric top with its axis of symmetry aligned along the 3-axis and with angular momentum  $K$  about this axis. Let  $\mathcal{O}_K$  be the coadjoint orbit containing  $\rho_0^{(K)}$ . When  $K \neq 0$ , the density  $\rho_0^{(K)}$  is no longer invariant under rotations through  $\pi$  about an axis perpendicular to the symmetry axis and  $\mathcal{O}_K$  becomes symplectomorphic to  $T^*(SO(2) \backslash SO(3))$ ; as a manifold,  $\mathcal{O}_K$  remains four dimensional.

**7.2.2. The classical dynamics of a symmetric top.** The classical dynamics of a symmetric top illustrate the advantages of working algebraically with observables rather than coordinates and of considering the component of angular momentum  $K$  about the symmetry axis as a gauge degree of freedom.

Suppose the classical Hamiltonian for a symmetric top is given by the standard function

$$\mathcal{H} = \frac{1}{2} \sum_{mn} \mathcal{L}_m \mathfrak{S}_{mn}^{-1} \mathcal{L}_n. \quad (120)$$

Because  $\mathcal{H}$  is rotationally invariant, the square of the angular momentum  $\mathcal{L}^2$  is a constant of the motion. And, for a symmetric top, the component  $K$  of the angular momentum along the symmetry axis is also a constant of the motion. Thus, in the principal axes frame of the rotor, the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2\mathfrak{S}_1} (\mathcal{L}_1^2 + \mathcal{L}_2^2) + \frac{1}{2\mathfrak{S}_3} K^2 = \frac{1}{2\mathfrak{S}_1} \mathcal{L}^2 + \text{constant} \quad (121)$$

where

$$\mathcal{L}^2 = \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_3^2. \quad (122)$$

Although derived in the principal axes frame, these expressions of  $\mathcal{H}$  and  $\mathcal{L}^2$  are valid in any reference frame, albeit with  $\mathfrak{S}_1$  and  $K$  regarded as numerical constants.

Now, because the phase space of a symmetric top is of dimension four, the motion of the rotor is characterized by the time evolution of any four linearly-independent observables, e.g., the components  $\{\mathcal{L}_1, \mathcal{L}_2, \mathfrak{S}_{13}, \mathfrak{S}_{23}\}$  of  $\mathcal{L}$  and  $\mathfrak{S}$  relative to the space-fixed axes. The time evolution of these observables is then given by solution of the equations of motion

$$\dot{\mathfrak{S}}_{i3} = \frac{1}{2\mathfrak{S}_1} \{\mathfrak{S}_{i3}, \mathcal{L}^2\} \quad \dot{\mathcal{L}}_i = \frac{1}{2\mathfrak{S}_1} \{\mathcal{L}_i, \mathcal{L}^2\} = 0 \quad i = 1, 2. \quad (123)$$

As expected, these equations confirm that each component of the angular momentum is conserved.

Suppose that the angular momentum has magnitude  $L$  and is aligned along the space-fixed 3-axis. Then the time evolution of the top is given by

$$\dot{\mathfrak{S}}_{13} = -\frac{L}{\mathfrak{S}_1} \mathfrak{S}_{23} \quad \dot{\mathfrak{S}}_{23} = \frac{L}{\mathfrak{S}_1} \mathfrak{S}_{13}. \quad (124)$$

These are the equations of a simple two-dimensional harmonic oscillator of frequency  $L/\mathfrak{S}_1$ . Thus, the top precesses about the 3-axis with this angular frequency. Note, however, that if the angular momentum lies along the symmetry axes of the symmetric top, then the symmetry axis coincides with the space-fixed 3-axis. And, since the symmetry axis is a principal axis of the inertia tensor, it then follows that  $\mathfrak{S}_{12} = \mathfrak{S}_{13} = 0$  and the top simply spins in the expected way, without precession, with angular momentum  $K = L$  about its symmetry axis.

**7.2.3. Semi-classical representations of the symmetric top.** The intrinsic degrees of freedom of a symmetric top are quantized in a semi-classical representation by replacing the classical phase space  $[\mathbb{R}^4]D_\infty \backslash [\mathbb{R}^6]SO(3) \simeq T^*(D_\infty \backslash SO(3))$  by a fibre bundle associated with the principal  $[\mathbb{R}^6]SO(3) \rightarrow [\mathbb{R}^4]D_\infty \backslash [\mathbb{R}^6]SO(3)$  bundle by a unitary irrep  $M$  of the isotropy subgroup  $[\mathbb{R}^4]D_\infty$ . Such semi-classical representations can be derived from an abstract unitary representation  $T$  of the RMG on a Hilbert space  $\mathbb{H}$  as follows.

Let  $|\xi_K\rangle = |K\rangle \in \mathbb{H}$  be a normalized state that satisfies equation (119) with  $\mathfrak{S}_1 = \mathfrak{S}_2 \neq \mathfrak{S}_3$ , and let  $|\xi_{\bar{K}}\rangle$  be defined by

$$|\xi_{\bar{K}}\rangle = |\bar{K}\rangle = T\left(e^{\frac{i}{\hbar}\pi L_2}\right)|K\rangle. \quad (125)$$

Let

$$E = |K\rangle\langle\xi_K| + |\bar{K}\rangle\langle\xi_{\bar{K}}| \quad (126)$$

be the natural embedding of the subspace  $U$ , spanned by the states  $\{|\xi_K\rangle, |\xi_{\bar{K}}\rangle\}$ , in  $\mathbb{H}$ , and let

$$\Pi = |\xi_K\rangle\langle K| + |\xi_{\bar{K}}\rangle\langle \bar{K}| \tag{127}$$

be the corresponding  $\mathbb{H} \rightarrow U$  projection operator. Together,  $E$  and  $\Pi$  define a semi-classical density  $\hat{\rho}^{(K)}(A) = \Pi T(A) E$  that (for  $K \neq 1/2$ ) satisfies

$$\begin{aligned} \hat{\rho}^{(K)}(L_i) &= 0 & i = 1, 2 \\ \hat{\rho}^{(K)}(L_3) &= K (|\xi_K\rangle\langle \xi_K| - |\xi_{\bar{K}}\rangle\langle \xi_{\bar{K}}|) \equiv \hat{S} \\ \hat{\rho}^{(K)}(I_{ij}) &= \delta_{ij} \mathfrak{S}_i (|\xi_K\rangle\langle \xi_K| + |\xi_{\bar{K}}\rangle\langle \xi_{\bar{K}}|) = \delta_{ij} \mathfrak{S}_i \hat{\mathcal{I}} \end{aligned} \tag{128}$$

with  $\hat{\mathcal{I}}$  the identity operator on  $U$ .

The subalgebra  $\mathfrak{h}$  of the RMA,  $\mathfrak{g} = [\mathbb{R}^6]so(3)$ , defined by

$$\mathfrak{h} = \{A \in \mathfrak{g} | \hat{\rho}([A, B]) = [\hat{\rho}(A), \hat{\rho}(B)], \forall B \in \mathfrak{g}\} \tag{129}$$

is the Lie algebra  $[\mathbb{R}^4]so(2)$ . Moreover, the restriction of  $\hat{\rho}$  to  $\mathfrak{h} \subset \mathfrak{g}$  is a reducible representation  $M$  for which

$$\begin{aligned} M(L_3)|\xi_K\rangle &= K|\xi_K\rangle & M(L_3)|\xi_{\bar{K}}\rangle &= -K|\xi_{\bar{K}}\rangle \\ M(I_{ii})|\xi_K\rangle &= \mathfrak{S}_i|\xi_K\rangle & M(I_{ii})|\xi_{\bar{K}}\rangle &= \mathfrak{S}_i|\xi_{\bar{K}}\rangle & i = 1, 2, 3 \\ M(I_{12})|\xi_K\rangle &= M(I_{12})|\xi_{\bar{K}}\rangle = 0. \end{aligned} \tag{130}$$

Note that, unless  $U \subset \mathbb{H}$  happens to be an  $\mathfrak{h}$ -invariant subspace, this representation of  $\mathfrak{h}$  is not a subrepresentation of the restriction of  $T$  to  $\mathfrak{h} \subset \mathfrak{g}$ . It is an example of an embedded representation, as discussed in section 3. Nevertheless, it integrates to a reducible (and generally projective) unitary irrep  $M$  of  $[\mathbb{R}^4]SO(2)$  which extends to an irreducible unitary irrep of  $[\mathbb{R}^4]D_\infty$  with, for example,

$$\begin{aligned} M(e^{-\frac{i}{\hbar}\theta L_3})|\xi_K\rangle &= e^{-iK\theta}|\xi_K\rangle & M(e^{-\frac{i}{\hbar}\theta L_3})|\xi_{\bar{K}}\rangle &= e^{iK\theta}|\xi_{\bar{K}}\rangle \\ M(e^{-\frac{i}{\hbar}\pi L_2})|\xi_K\rangle &= (-1)^{2K}|\xi_{\bar{K}}\rangle & M(e^{-\frac{i}{\hbar}\pi L_2})|\xi_{\bar{K}}\rangle &= |\xi_K\rangle. \end{aligned} \tag{131}$$

The operator  $\hat{\rho}$  also defines a semi-classical representation of any element  $A$  in the RMA by an operator-valued function  $\hat{A}$  over the RMG with values

$$\hat{A}(g) = \hat{\rho}(A(g)) \quad g \in [\mathbb{R}^6]SO(3) \tag{132}$$

where  $A(g) = \text{Ad}_g(A)$ . These functions satisfy the  $[\mathbb{R}^4]D_\infty$ -equivariance condition

$$\hat{\rho}_{hg} = M(h)\hat{\rho}_g M(h^{-1}) \quad \forall h \in [\mathbb{R}^4]D_\infty \tag{133}$$

and have Poisson brackets

$$\{\hat{A}, \hat{B}\}(g) = -\frac{i}{\hbar} \hat{\rho}([A(g), B(g)]) \quad \forall g \in [\mathbb{R}^6]SO(3). \tag{134}$$

**7.2.4. Quantization of a symmetric top.** There is a natural polarization for any cotangent bundle and, as a result, the full quantization of a rotor is simpler than prequantization. We therefore bypass prequantization and proceed directly to quantization by constructing an appropriate unitary irrep of the RMG. The natural polarization for the symmetric top is defined by starting with a representation  $M$  of the isotropy subgroup  $[\mathbb{R}^4]D_\infty$  for the phase space of a symmetric top and extending it to a representation  $\tilde{M}$  of  $[\mathbb{R}^6]D_\infty$ . Such a representation is defined as

$$\tilde{M}(Q, \omega) = e^{-\frac{i}{\hbar} Q \cdot \bar{\mathfrak{S}}} M(\omega) \quad Q \in \mathbb{R}^6 \quad \omega \in D_\infty, \tag{135}$$

where  $\bar{\mathfrak{S}}$  is the diagonal matrix whose entries are the principal moments of inertia  $(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3)$  of the rotor, and  $Q \cdot \bar{\mathfrak{S}} = \sum_{ij} Q_{ij} \bar{\mathfrak{S}}_{ij} = \sum_i Q_{ii} \mathfrak{S}_i$ .

Note, however, that for the semi-classical representation defined by  $M$  to be quantizable, it is necessary that  $2K$  should be an integer. Otherwise the representation of  $SO(2)$  labelled by  $K$  will not be contained in any representation of  $SO(3)$ . If  $2K$  is odd, then  $M$  is contained in a projective (spinor) representation of  $SO(3)$ , i.e. a true representation of  $SU(2)$ , the double cover of  $SO(3)$ . Thus, to avoid the subtleties associated with projective representations, it will be convenient in the following to regard  $\tilde{M}$  as a true irrep of  $[\mathbb{R}^4]D_\infty^d$ , the double cover of  $[\mathbb{R}^4]D_\infty$ , and require that it be contained in some unitary representation of  $[\mathbb{R}^4]SU(2)$ .

Let  $U$  be the carrier space for the irrep  $\tilde{M}$  of  $[\mathbb{R}^6]D_\infty^d$ . Now, we no longer require  $U$  to be a subspace of the Hilbert space  $\mathbb{H}$  for the abstract representation  $T$  of the RMG. Instead, an irrep of the RMG is induced in VCS theory by defining an  $[\mathbb{R}^6]D_\infty^d$ -intertwining operator  $\Pi : \mathbb{H}_D \rightarrow U$  from a suitably defined dense subspace  $\mathbb{H}_D \subset \mathbb{H}$  to  $U$ , such that

$$\Pi T(Q, \omega) = e^{-\frac{i}{\hbar} Q \cdot \vec{\omega}} M(\omega) \Pi \quad Q \in \mathbb{R}^6 \quad \omega \in D_\infty^d. \quad (136)$$

VCS wavefunctions are then defined initially as vector-valued functions over  $[\mathbb{R}^6]SU(2)$  with values in  $U$  given by

$$\Psi(Q, \Omega) = \Pi T(Q, \Omega) |\Psi\rangle \quad |\Psi\rangle \in \mathbb{H}_D. \quad (137)$$

Because of the constraint condition (136), these functions satisfy

$$\Psi(Q, \Omega) = e^{-\frac{i}{\hbar} Q \cdot \vec{\Omega}} \Pi R(\Omega) |\Psi\rangle \quad Q \in \mathbb{R}^6 \quad \Omega \in SU(2) \quad (138)$$

where  $R(\Omega) = T(0, \Omega)$  is the restriction of the representation  $T$  to  $SU(2)$ . Thus, it is sufficient to define VCS wavefunctions as the vector-valued functions over  $SU(2)$

$$\psi(\Omega) = \Pi R(\Omega) |\Psi\rangle \quad |\Psi\rangle \in \mathbb{H}_D \quad \Omega \in SU(2) \quad (139)$$

which satisfy the condition

$$\psi(\omega\Omega) = M(\omega)\psi(\Omega) \quad \forall \omega \in D_\infty^d. \quad (140)$$

The VCS representation of the RMG is now defined on these wavefunctions by

$$[\Gamma(Q, \Omega)\psi](\Omega') = \Pi R(\Omega') T(Q, \Omega) |\Psi\rangle = \Pi T(\Omega' Q \tilde{\Omega}', \Omega') |\Psi\rangle \quad (141)$$

which gives

$$[\Gamma(Q, \Omega)\psi](\Omega') = e^{-\frac{i}{\hbar} (\Omega' Q \tilde{\Omega}') \cdot \vec{\Omega}} \Psi(\Omega' \Omega). \quad (142)$$

An explicit construction of the Hilbert space for this VCS representation is presented as follows. First observe from equation (142) that a reducible representation  $T$  of the RMG is defined on the Hilbert space  $\mathbb{H} = \mathcal{L}^2(SU(2))$  by

$$[T(Q, \Omega)\psi](\Omega') = e^{-\frac{i}{\hbar} (\Omega' Q \tilde{\Omega}') \cdot \vec{\Omega}} \Psi(\Omega' \Omega). \quad (143)$$

Now, by the Peter–Weyl theorem, an orthonormal basis for  $\mathcal{L}^2(SU(2))$  is given by the  $SU(2)$  Wigner functions

$$\Phi_{NJM} = \sqrt{\frac{2J+1}{8\pi^2}} \mathcal{D}_{NM}^J \quad (144)$$

where  $2J$  is a positive or zero integer and  $M$  and  $N$  run from  $-J$  to  $+J$  in integer steps. Let  $\{|NJM\rangle\}$  denote the vector in  $\mathbb{H}$  with wavefunction  $\Phi_{NJM}$  and let  $\mathbb{H}_D$  denote the dense subspace of finite linear combinations of these basis vectors. Now let  $\langle K|$  and  $\langle \bar{K}|$  denote the functionals on  $\mathbb{H}_D$  for which

$$\langle K|NJM\rangle = \sqrt{\frac{2J+1}{8\pi^2}} \delta_{NK} \delta_{MK} \quad \langle \bar{K}|NJM\rangle = (-1)^{J+K} \sqrt{\frac{2J+1}{8\pi^2}} \delta_{NK} \delta_{M,-K}. \quad (145)$$

Let  $\Pi$  denote the operator

$$\Pi = \frac{1}{\sqrt{2}}(\xi_K \langle K | + \xi_{\bar{K}} \langle \bar{K} |) \quad (146)$$

that maps  $\mathbb{H}_D \rightarrow U$ , where  $\{\xi_K, \xi_{\bar{K}}\}$  is the basis for  $U$  as defined above with  $2K$  a fixed positive integer. This operator satisfies the intertwining condition

$$\Pi R(\omega) = M(\omega) \Pi \quad \forall \omega \in D_\infty^d \quad (147)$$

and defines a basis  $\{\psi_{KJM}\}$  for a Hilbert space  $\mathcal{H}^K$  of coherent state wavefunctions, having values

$$\psi_{KJM}(\Omega) = \Pi R(\Omega) |KJM\rangle = \sqrt{\frac{2J+1}{16\pi^2}} [\xi_K \mathcal{D}_{KM}^J(\Omega) + (-1)^{J+K} \xi_{\bar{K}} \mathcal{D}_{-K,M}^J(\Omega)]. \quad (148)$$

This basis is seen to be orthonormal relative to the natural  $U \otimes \mathcal{L}^2(SU(2))$  inner product. It is the standard basis of rotor model wavefunctions used in nuclear physics [18, 20, 21].

The map  $\mathbb{H} \rightarrow \mathcal{H}^K$ , defined by equation (148), shows that  $\mathcal{H}^K$  is isomorphic to a subspace of  $\mathbb{H}$ . From the theory of induced representations, it is known that this subspace is irreducible. Thus, the irrep  $\tilde{M}$  of the subgroup  $[\mathbb{R}^6]D_\infty^6$  uniquely defines an irreducible representation of the RMG and its Lie algebra RMA and, hence, a quantization of the symmetric top model.

## 8. Concluding remarks

Coherent state representation theory has its most general expression in vector coherent state (VCS) theory. This theory is now highly developed as a practical theory of induced representations. It encompasses virtually all the standard inducing constructions. In addition, it facilitates the construction of orthonormal bases for irreducible representations and provides practical algorithms for the computation of the matrix coefficients for the irreps of model spectrum generating algebras. By having the flexibility to induce irreps of a group  $G$  from a multidimensional irrep of a subgroup  $H \subset G$ , VCS theory has a huge practical advantage over its scalar counterpart. It has been used to construct irreps of representative examples of all the classical Lie algebras and has been applied widely to models in nuclear physics (cf [7, 13]).

The relationship of geometric quantization to the theory of induced representations is surely well understood by experts in the two fields. However, the new insights and simplifications that can be brought to the practical application of both theories by VCS theory is not known. We hope to have shown in this paper that, by understanding the relationships between the three theories when they are expressed in a common language, it becomes possible to exploit their complementary features to greatest advantage.

Already some new perspectives and new approaches to old physical problems are suggested by the unified approach to quantization presented here. An important advance in modern physics has been the development of Abelian and non-Abelian gauge theory. It has long been known that (often hidden) intrinsic motions can have a profound effect on the dynamics of a system. A well-known example of this is the precessional motion of a symmetric top that is spinning in a way that may not be directly observable about its symmetry axis.

The VCS methods outlined in this paper suggest ways to select physically and mathematically relevant intrinsic degrees of freedom and express their influence on the complementary extrinsic dynamics in terms of gauge potentials. For example, it might be appropriate to regard the fast and slow degrees of freedom of a many-body system, for which the Born–Oppenheimer approximation [22] applies, as being intrinsic and collective, respectively. A model description of the scattering of two nuclei for which the motions of their



centres of mass are *adiabatic* in comparison to their intrinsic degrees of freedom would be a candidate for such a separation. Quantum optics also provides examples, such as parametric down conversion (PDC), that possess useful algebraic models [23]; in the standard description of PDC, strong coherent beams are treated classically while down-converted photon pairs must be described quantally. In physical systems where some degrees of freedom tend to behave in a manifestly quantal way while others are essentially classical, the proposals given in section 3 for deriving partial quantizations have the potential for providing systematic ways of modelling such systems. In particular, methods are given for constructing semi-classical models in which the intrinsic (gauge) degrees of freedom are quantized but the extrinsic dynamics are treated classically. Such models provide a valuable framework for interpreting the widespread phenomena of collective subdynamics observed in systems whose collective motions are adiabatic but whose intrinsic excitations, when treated quantally, lie at relatively high energies.

A problem of considerable interest is the description of vortices in quantum fluids. This problem has been related to the notoriously difficult task of constructing the unitary irreps of infinite-dimensional groups of diffeomorphisms [24]. The potential relevance of the methods proposed in this paper to this problem are indicated by the following observations. A model of linear hydrodynamic flows in nuclei (the so-called CM(3) model) has been quantized both by induced representation methods [25] and geometric quantization [26]. These quantizations are characterized by quantized vortex spins which are naturally regarded as intrinsic  $SU(2)$  degrees of freedom. The second observation is that VCS methods have been successfully applied to the infinite-dimensional affine Lie algebra  $\widehat{sl}(2)$  [27].

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### Appendix. The covariant derivative and curvature tensor

**Claim.** Let  $B$  be a vector bundle with typical fibre  $U$  associated with a principal  $G \rightarrow H \backslash G$  bundle by a unitary representation  $M$  of  $H \subset G$ . Let  $\hat{\rho}$  be an  $H$ -equivariant  $\mathfrak{g} \rightarrow GL(U)$  map having the property that it maps the subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  to the representation  $M$ , i.e.  $\hat{\rho}(A) = M(A)$  for  $A \in \mathfrak{h}$  (cf text for details). Define

$$i\hbar[\nabla_A \Psi](g) = \Psi(A(g)g) - \hat{\rho}(A(g))\Psi(g) \quad (\text{A.1})$$

where  $A(g) = \text{Ad}_g(A)$ ,  $\Psi(Ag)$  is defined for any  $A \in \mathfrak{g}$  by equation (43), and  $\Psi$  is any section of  $B$ , i.e. it satisfies the identity

$$\Psi(hg) = M(h)\Psi(g) \quad \forall h \in H. \quad (\text{A.2})$$

Then  $\nabla_A$  is identical to

$$\nabla_{X_{\hat{A}}} = X_{\hat{A}} + \frac{i}{\hbar}\hat{\theta}(X_{\hat{A}}) \quad (\text{A.3})$$

the covariant derivative in the direction of the Hamiltonian vector field  $X_{\hat{A}}$  over  $H \backslash G$  generated by the vector-valued function  $\hat{A}(g) = \hat{\rho}(A(g))$ , where  $\hat{\theta}$  is a symplectic connection (one-form) for  $B$ .

**Proof.** A choice of gauge is defined by the expansion

$$A(g) = \sum_i A^i(g)A_i + \sum_\nu A^\nu(g)A_\nu \quad (\text{A.4})$$

where  $\{A_i\}$  is a basis for  $\mathfrak{h}$  and  $\{A_\nu\}$  completes a basis for  $\mathfrak{g}$ . Using the identity  $\Psi(A_i g) = \hat{\rho}(A_i)\Psi(g)$ , definition (A.1) gives

$$i\hbar[\nabla_A \Psi](g) = \sum_\nu A^\nu(g)(\Psi(A_\nu g) - \hat{\rho}(A_\nu)\Psi(g)). \tag{A.5}$$

Now, if  $g(x) = e^{X(x)}g$ , with  $X(x) = -\frac{i}{\hbar} \sum_\mu x^\mu A_\mu$ , then as shown in the appendix to [1],

$$i\hbar \frac{\partial}{\partial x^\nu} \Psi(g(x)) = \Psi(A_\nu(x)g(x)) \tag{A.6}$$

where

$$A_\nu(x) = -i\hbar e^{X(x)} \frac{\partial}{\partial x^\nu} e^{-X(x)} = A_\nu + \frac{1}{2!}[X(x), A_\nu] + \frac{1}{3!}[X(x), [X(x), A_\nu]] + \dots \tag{A.7}$$

Therefore, if  $A_\nu(x)$  is expanded

$$A_\nu(x) = \sum_\mu \Lambda_\nu^\mu(x) A_\mu + \sum_i \lambda_\nu^i(x) A_i \tag{A.8}$$

then

$$\Psi(A_\nu g(x)) = i\hbar \sum_\mu \bar{\Lambda}_\nu^\mu(x) \left( \frac{\partial}{\partial x^\mu} + \frac{i}{\hbar} \lambda_\mu^i(x) \hat{\rho}(A_i) \right) \Psi(g(x)) \tag{A.9}$$

where  $\bar{\Lambda}$  is the inverse of the matrix  $\Lambda$ . It follows from equation (A.5) that

$$[\nabla_A \Psi](g(x)) = \sum_\nu A^\nu(g(x)) \bar{\Lambda}_\nu^\mu(x) \left( \frac{\partial}{\partial x^\mu} + \frac{i}{\hbar} \hat{\theta}_\mu(x) \right) \Psi(g(x)) \tag{A.10}$$

where

$$\hat{\theta}_\mu(x) = \sum_\nu \Lambda_\mu^\nu(x) \hat{\rho}(A_\nu) + \sum_i \lambda_\mu^i(x) \hat{\rho}(A_i) = \hat{\rho}(A_\mu(x)). \tag{A.11}$$

Thus, if we regard  $\hat{\theta}_\mu(x)$  as the component  $\hat{\theta}_{g(x)}(\partial/\partial x^\mu)$  of a one-form  $\hat{\theta}$ , defined at  $g(x)$  by

$$\hat{\theta}_{g(x)} = \sum_\mu \hat{\theta}_\mu(x) dx^\mu \tag{A.12}$$

and define the Hamiltonian vector field  $X_{\hat{A}}$  by

$$[X_{\hat{A}} \Psi](g(x)) = \sum_\nu A^\nu(g) \bar{\Lambda}_\nu^\mu(x) \frac{\partial}{\partial x^\mu} \Psi(g(x)) \tag{A.13}$$

then

$$\nabla_A = X_{\hat{A}} + \frac{i}{\hbar} \hat{\theta}(X_{\hat{A}}) \tag{A.14}$$

as claimed.

To check that  $\hat{\theta}$  is a symplectic connection, we now derive the curvature of the connection one-form  $\hat{\theta}$ . Consider first the standard exterior derivative of  $\hat{\theta}$  given by

$$d\hat{\theta}_{g(x)} = \sum_{\mu\nu} \frac{\partial \hat{\theta}_\nu(x)}{\partial x^\mu} dx^\mu \wedge dx^\nu. \tag{A.15}$$

From definition (A.11) of  $\hat{\theta}_\nu(x)$ , and with  $A_\nu(x)$  expressed by equation (A.7),

$$\frac{\partial \hat{\theta}_\nu(x)}{\partial x^\mu} = -i\hbar \frac{\partial}{\partial x^\mu} \hat{\rho} \left( e^{X(x)} \frac{\partial}{\partial x^\nu} e^{-X(x)} \right). \tag{A.16}$$

Then, with the help of the identities (see [1])

$$i\hbar \frac{\partial e^{X(x)}}{\partial x^\nu} = A_\nu(x) e^{X(x)} = e^{X(x)} A_\nu(-x) \quad (\text{A.17})$$

$$i\hbar \frac{\partial e^{-X(x)}}{\partial x^\nu} = -A_\nu(-x) e^{-X(x)} = -e^{-X(x)} A_\nu(x) \quad (\text{A.18})$$

we obtain

$$d\hat{\theta}_{g(x)} \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\mu} \right) = \frac{\partial \hat{\theta}_\nu(x)}{\partial x^\mu} = -\frac{i}{\hbar} \hat{\rho}([A_\mu(x), A_\nu(x)]). \quad (\text{A.19})$$

Thus, from the expansion of  $A_\nu(x)$  given by equation (A.8), and recalling that

$$\hat{\rho}([A, B]) = [\hat{\rho}(A), \hat{\rho}(B)] \quad A \in \mathfrak{h} \quad B \in \mathfrak{g} \quad (\text{A.20})$$

we derive

$$d\hat{\theta}_{g(x)} \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\mu} \right) = \sum_{\mu'\nu'} \Lambda_{\mu'}^{\mu'}(x) \hat{\Omega}_{\mu'\nu'} \Lambda_{\nu'}^{\nu'}(x) + [\hat{\theta}_\mu(x), \hat{\theta}_\nu(x)]. \quad (\text{A.21})$$

with

$$\hat{\Omega}_{\mu\nu} = -\frac{i}{\hbar} (\hat{\rho}([A_\mu, A_\nu]) - [\hat{\rho}(A_\mu), \hat{\rho}(A_\nu)]). \quad (\text{A.22})$$

It follows that, for the vector fields defined by equation (A.13),

$$d\hat{\theta}_{g(x)}(X_{\hat{A}}, X_{\hat{B}}) = \sum_{\mu\nu} A^\mu(g(x)) \hat{\Omega}_{\mu\nu} B^\nu(g(x)) + [\hat{\theta}_{g(x)}(X_{\hat{A}}), \hat{\theta}_{g(x)}(X_{\hat{B}})]. \quad (\text{A.23})$$

Hence we derive the general expression for the curvature tensor

$$\hat{\Omega}(X_{\hat{A}}, X_{\hat{B}}) = d\hat{\theta}(X_{\hat{A}}, X_{\hat{B}}) - [\hat{\theta}(X_{\hat{A}}), \hat{\theta}(X_{\hat{B}})]. \quad (\text{A.24})$$

□

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